4-th Project Workshop

Ageing and Maintenance in Reliability: Modelling and Statistical Inference

The Gamma process and its generalizations for describing age- and/or state-dependent degradation phenomena

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For several highly reliable products, **it is often difficult to evaluate their reliability** using classical methods based on failure times due to the **very low number of failures** that are generally observed.

Fortunately, many such products have **performance characteristics whose decay or degradation** over the operating time can be related to their **reliability**.
Thus, if the **degradation phenomenon** of these products is observed and adequately modelled, it is possible to accurately evaluate their reliability.

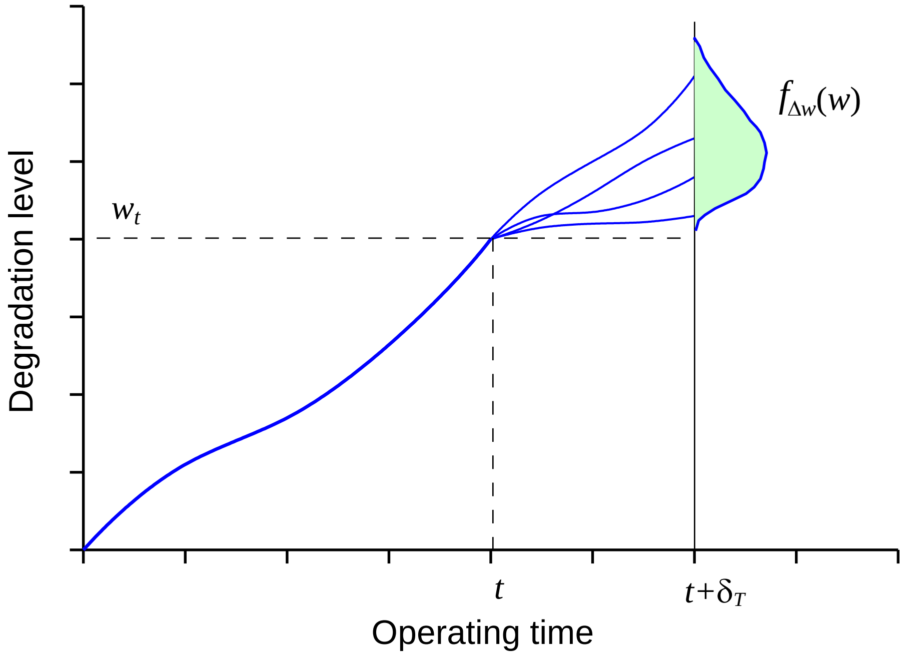
In addition, an accurate **modelization of the degradation phenomenon** allows one to predict the **remaining life**, and then **we can plan a condition-based maintenance policy**, which can be more effective both than an age-based preventive maintenance and than a corrective maintenance policy.

Of course, if the **degradation phenomenon is incorrectly modelled**,
the reliability evaluation, the lifetime prediction and the maintenance optimization
can result in inaccurate.

Let $\{W(t); t \geq 0\}$ denote **an increasing, continuous-time, stochastic process**.

In general, when a unit is subject to a degradation phenomenon, the **probability distribution of the degradation increment** $\Delta W(t,\delta_T) \equiv W(t+\delta_T) - W(t)$ over the future time interval $(t, t+\delta_T)$ **depends on the whole history** of the degradation process up to the current time t.

When $\Delta W(t,\delta_T)$ is assumed to depend only on the present and not on the past history, then **the process is said to be Markovian**. Within the **Markovian processes**, **we can distinguish the cases** where $\Delta W(t,\delta_T)$ depends (functionally or stochastically):



a) only on the interval length δ_T (*homogeneous or stationary process* with independent increments)

b) on δ_T and on the current age t (*age-dependent process* with independent increments)

c) on δ_T and on the current state w_t (*state-dependent process* with dependent increments)

d) on δ_T, on the current age t and state w_t (*age- and state-dependent process* with dependent increments)

When the degradation growth of an item **depends only on its age** and not on the current degradation, a **widely accepted model** is the **non-stationary Gamma process**.

The non-stationary Gamma process $\{W(t), t \geq 0\}$ has the following properties:

(a) the **increments** $\Delta W(t,\delta_T) \equiv W(t+\delta_T) - W(t)$ $(t,\delta_T > 0)$ over disjoint time intervals $(t, t+\delta_T)$ are **independent random variables**, and

(b) each **increment** $\Delta W(t,\delta_T)$, for all t and δ_T, follows **a Gamma distribution** with constant scale parameter $\beta > 0$ and (possibly) time-varying shape parameter $\Delta\eta(t,\delta_T) = \eta(t+\delta_T) - \eta(t)$:

$$f_{\Delta W(t,\delta_T)}(w|t) = \frac{(w/\beta)^{\Delta\eta(t,\delta_T)-1}}{\beta\ \Gamma[\Delta\eta(t,\delta_T)]} \exp(-w/\beta)\ , \quad w > 0 \tag{1}$$

- $\eta(\bullet)$ is a non-negative, monotone increasing function for $t \geq 0$ with $\eta(0) = 0$,
- $\Gamma(a) = \int_0^\infty z^{a-1} \exp(-z)\, dz$ is the complete Gamma function.

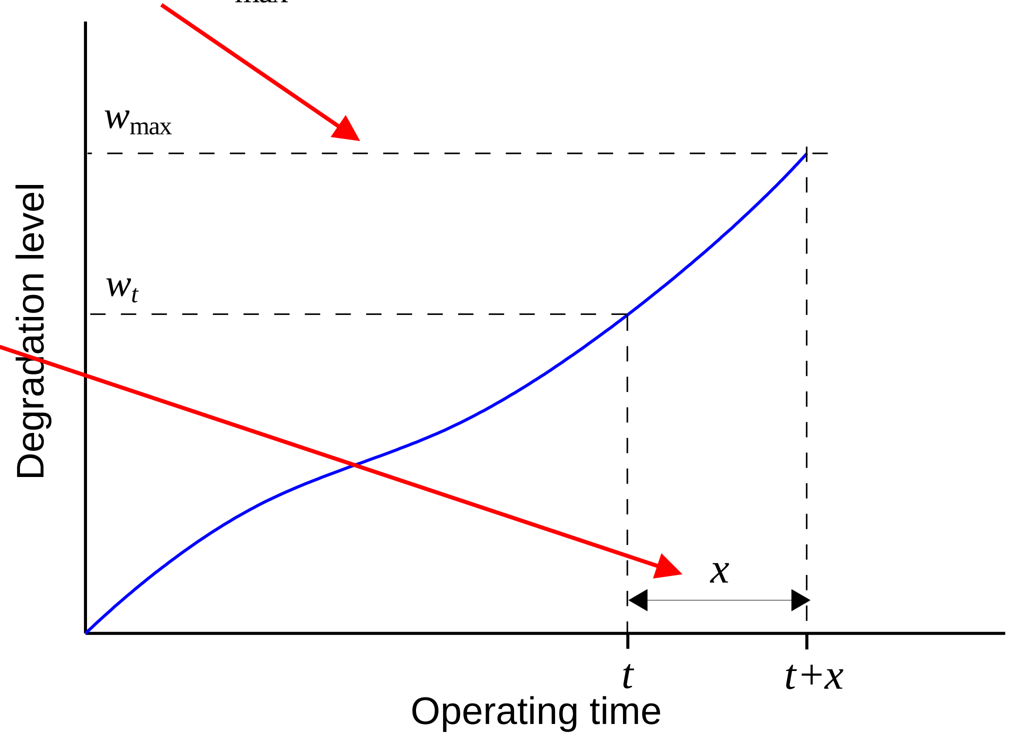
The unusual notation $f_{\Delta W(t,\delta_T)}(w|t)$ is here used to stress the "*functional*" dependence on the current age

If $\eta(t) \propto t$, the process is **homogeneous:** the increments distribution is independent on the age t.

A **unit** subject to the degradation process is generally assumed to **fail when its degradation level exceeds a threshold level** $w_{\max}$.

The **unit lifetime** is then defined as the operating time needed to exceed $w_{\max}$.

Hence, the **remaining lifetime** X of a unit which has not failed at the current time t is the extra time the unit spends to exceed the level $w_{\max}$, when starting from the current state $W(t) = w_t$.



The **residual reliability** is the probability that:

$$R_t(x|w_t) = \Pr\{\Delta W(t,x) \le w_{\max} - w_t \,|\, t, w_t\}$$

and, under the **Gamma process**, is given by:

$$R_t(x|w_t) = \int_0^{(w_{\max}-w_t)/\beta} \frac{z^{\Delta\eta(t,x)-1}}{\Gamma[\Delta\eta(t,x)]} \exp(-z)\,\mathrm{d}z \,. \qquad (2)$$

It depends on the current state w_t only through the difference $w_{\max} - w_t$.

The **mean** and **variance** of the degradation $W(t)$ at time t of the Gamma process are:

$$\mathrm{E}\{W(t)\} = \eta(t) \cdot \beta \qquad \text{and} \qquad \mathrm{V}\{W(t)\} = \eta(t) \cdot \beta^2 . \tag{3}$$

Thus, the **variance-to-mean ratio** $\mathrm{V}\{W(t)\}/\mathrm{E}\{W(t)\}$ equals the scale parameter β and **does not depend on time** t.

Note that even in the **Inverse Gaussian process** (the other commonly used process with positive increments) the ratio $\mathrm{V}\{W(t)\}/\mathrm{E}\{W(t)\}$ is constant with t

The independent increments property and the assumption of a constant scale parameter β make the subsequent mathematics quite tractable, and **have encouraged the widespread use of the Gamma process**, even **without performing the necessary checks on the adequacy** of the Gamma process to the degradation phenomenon to analyze.

For example, even if the degradation process is purely age-dependent, the **non-stationary Gamma process is not a proper choice** when, in absence of heterogeneity and/or measurement errors, there is **empirical evidence that the variance-to-mean ratio is not constant**.

A degradation model where the **variance-to-mean ratio is not restricted to be constant** is the **Extended Gamma process**, which is defined by:

$$X(t) = \int_0^t \xi(s)\,\mathrm{d}Z(s) \quad , \tag{4}$$

- $\{Z(s), s \geq 0\}$ is a Gamma process with shape parameter $\eta(s)$ and unit scale parameter,
- $\xi(s)$ is a positive, right continuous, real-valued function,
- the integration is done with respect to the sample paths of $Z(s)$.

The **Extended Gamma process** $\{X(t), t \geq 0\}$ **has independent increments**, and when $\xi(s)$ is a constant for all $s \geq 0$, it reduces to the Gamma process.

The **mean** and **variance** of $W(t)$ are given by

$$\mathrm{E}\{X(t)\} = \int_0^t \xi(s)\,\mathrm{d}\eta(s) \quad \text{and} \quad \mathrm{V}\{X(t)\} = \int_0^t \xi^2(s)\,\mathrm{d}\eta(s) \ , \tag{5}$$

and hence the **variance-to-mean ratio is no longer necessarily constant**.

Unfortunately, the distribution function of $X(t)$ is **very difficult to treat mathematically**.

A way to overcome the mathematical difficulty of the Extended Gamma process is **to discretize the time axis** t with (small) time increments of length h.

The time-discretized model $W(t)$ (called the **time-discrete Extended Gamma process**) is based on the following assumptions:

1. the **increments** over disjoint time intervals are **independent random variables**,
2. each **"*elementary*" increment** $\Delta W(t,h) \equiv W(t+h) - W(t)$ over a very small time interval $(t, t+\delta_T)$ of length h, **follows a Gamma distribution** with mean and variance:

$$E\{\Delta W(t,h)\,|\,t\} = \varphi(t+h) - \varphi(t) \qquad V\{\Delta W(t,h)\,|\,t\} = \psi(t+h) - \psi(t), \tag{6}$$

where $\varphi(\bullet)$ and $\psi(\bullet)$ are differentiable, monotone increasing functions, with $\varphi(0) = 0$ and $\psi(0) = 0$.

Then, under the **time-discrete Extended Gamma process**:

a) **the Gamma distribution** of each **increment** $\Delta W(t,h)$ has time-varying scale and shape parameters:

$$\beta(t)=\frac{\mathrm{V}\{\Delta W(t,h)\}}{\mathrm{E}\{\Delta W(t,h)\}}=\frac{\psi(t+h)-\psi(t)}{\varphi(t+h)-\varphi(t)} \quad \text{and} \quad \eta(t)=\frac{\mathrm{E}^2\{\Delta W(t,h)\}}{\mathrm{V}\{\Delta W(t,h)\}}=\frac{[\varphi(t+h)-\varphi(t)]^2}{\psi(t+h)-\psi(t)} \tag{7}$$

b) the **increment** $\Delta W(t,\nu\, h)\equiv W(t+\nu\, h)-W(t)$ $(\nu=2,3,\ldots)$ over the non-elementary time interval $(t,t+\nu\, h)$ of length $\nu\, h$ is the **sum of** ν **independent, but not identically distributed Gamma** r.v.:

$$\Delta W(t,\nu\, h)=\sum_{l=1}^{\nu}\Delta W(t+(l-1)\, h,h)=\sum_{l=1}^{\nu}\Delta W_l \quad , \tag{8}$$

and each Gamma r.v. ΔW_l has scale and shape parameters which depend on the age $t+l\, h$ at the beginning of the elementary time interval and on the length h:

$$\beta_l=\beta(t+l\, h)=\frac{\psi(t+l\, h)-\psi[t+(l-1)\, h]}{\varphi(t+l\, h)-\varphi[t+(l-1)\, h]} \quad \text{and} \quad \eta_l=\eta(t+l\, h)=\frac{\{\varphi(t+l\, h)-\varphi[t+(l-1)\, h]\}^2}{\psi(t+l\, h)-\psi[t+(l-1)\, h]} \tag{9}$$

c) the **mean** and **variance** of the degradation level $W(t)$ at time t are:

$$\mathrm{E}\{W(t)\}=\varphi(t) \quad \text{and} \quad \mathrm{V}\{W(t)\}=\psi(t) \ ,$$

and hence the **variance-to-mean ratio is not necessarily constant** with t.

As $h \to 0$, the **time-discrete Extended Gamma process tends to the Extended Gamma process** with time-varying scale parameter $\xi(t) = \psi'(t)/\varphi'(t)$ and with $\mathrm{d}\eta(t) = [\varphi'(t)]^2 / \psi'(t)\,\mathrm{d}t$.

Indeed, for such an **Extended Gamma process**, it results that $\mathrm{d}X(t)$ is a Gamma random variable with

$$\mathrm{E}\{\mathrm{d}X(t)|t\} = \xi(t)\,\mathrm{d}\eta(t) = \frac{\psi'(t)}{\varphi'(t)} \frac{[\varphi'(t)]^2}{\psi'(t)} \mathrm{d}t = \varphi'(t)\,\mathrm{d}t$$

$$\mathrm{V}\{\mathrm{d}X(t)|t\} = \xi^2(t)\,\mathrm{d}\eta(t) = \left[\frac{\psi'(t)}{\varphi'(t)}\right]^2 \frac{[\varphi'(t)]^2}{\psi'(t)} \mathrm{d}t = \psi'(t)\,\mathrm{d}t \;.$$

On the other hand, the **time-discrete Extended Gamma assumption** that the elementary increment $\Delta W(t,h)$ is a Gamma random variable yields:

$$\lim_{h\to 0} \mathrm{E}\{\Delta W(t,h)|t\} = \lim_{h\to 0} [\varphi(t+h) - \varphi(t)] = \varphi'(t)\,\mathrm{d}t$$

$$\lim_{h\to 0} \mathrm{V}\{\Delta W(t,h)|t\} = \lim_{h\to 0} [\psi(t+h) - \psi(h)] = \psi'(t)\,\mathrm{d}t \;.$$

Thus, as $h \to 0$, the increment $\Delta W(t,h)$ converges to $\mathrm{d}X(t)$ for each time t, and hence the proposed **time-discrete model** $\{W(t), t \geq 0\}$ **converges to the Extended Gamma process** $\{X(t), t \geq 0\}$.

Since the elementary increments ΔW_l are independent Gamma random variables with different scale parameters, the **exact density function of the degradation increment**

$$\Delta W(t, \nu h) = \sum_{l=1}^{\nu} \Delta W_l \tag{10}$$

can be obtained, for example, from the following **Gamma series**:

$$f_{\Delta W(t,\nu h)}(w|t) = \left[\prod_{l=1}^{\nu} \left(\frac{\beta_1}{\beta_l} \right)^{\eta_l} \right] \sum_{k=0}^{\infty} \frac{\delta_k \; w^{\rho+k-1}}{\beta_1^{\rho+k} \; \Gamma(\rho+k)} \exp(-w/\beta_1) \tag{11}$$

where: $\beta_1 = \min(\beta_l)$, $\rho = \sum_{l=1}^{\nu} \eta_l$, $\delta_0 = 1$, and the

coefficients δ_k $(k = 1, 2, \ldots)$ satisfy the recurrence relations: $\delta_{k+1} = \frac{1}{k+1} \sum_{s=1}^{k+1} \left[\sum_{l=1}^{\nu} \eta_l (1 - \beta_1/\beta_l)^s \right] \delta_{k+1-s}$, $k = 0, 1, \ldots$.

The **Gamma series** in (11) **is very convenient for computational purposes** because the coefficients δ_k $(k = 0, 1, \ldots)$ can be easily computed.

For practical purposes, **this sum is stopped** at the first $K+1$ terms of the series, where K is determined in order to attain a desired accuracy.

When all **the elementary increments** $\Delta W_l = \Delta W(t+(l-1)\,h,\ h)$ **are Gamma distributed with common scale parameter** $\beta_l \equiv \beta$ (so that the resulting degradation process is just Gamma), then:

$$\beta_1 = \beta \ , \quad \delta_0 = 1 \ , \quad \text{and} \quad \delta_k = 0 \ , \text{ for all } k \geq 1 \ .$$

As a consequence, the pdf of the increment $\Delta W(t, \nu \cdot h)$ over the non-elementary interval $(t, t+\nu\, h)$

$$f_{\Delta W(t,\nu\, h)}(w|t) = \left[\prod_{l=1}^{\nu}\left(\frac{\beta_1}{\beta_l}\right)^{\eta_l}\right]\sum_{k=0}^{\infty}\frac{\delta_k\ w^{\rho+k-1}}{\beta_1^{\rho+k}\ \Gamma(\rho+k)}\exp(-w/\beta_1) \qquad (12)$$

becomes exactly the Gamma density

$$f_{\Delta W(t,\nu\, h)}(w|t) = \frac{w^{\rho-1}}{\beta^{\rho}\ \Gamma(\rho)}\exp(-w/\beta) \qquad (13)$$

with shape parameter $\rho = \sum_{l=1}^{\nu}\eta_l = \eta(t+\nu\, h) - \eta(t) = \Delta\eta(t, \nu\, h)$.

Then, **the Gamma process** can be viewed as
a special case of the time-discrete Extended Gamma process.

Under the time-discrete Extended Gamma process, **the conditional residual reliability** $R_t(x_\nu \mid w_t)$ is a **non-increasing step function**, evaluated at the (discrete) time $x_\nu = \nu\, h$:

$$R_t(x_\nu \mid w_t) = \Pr\{\Delta W(t, \nu\, h) \le w_{\max} - w_t \mid t\} = \left[\prod_{l=1}^{\nu} \left(\frac{\beta_1}{\beta_l}\right)^{\eta_l}\right] \sum_{k=0}^{\infty} \delta_k \int_0^{(w_{\max} - w_t)/\beta_1} \frac{z^{\rho+k-1}}{\Gamma(\rho+k)} \exp(-z)\, \mathrm{d}z \;. \quad (14)$$

$R_t(x_\nu \mid w_t)$ **depends on** w_t **only through the difference** $w_{\max} - w_t$.

From the reliability expression (14), we can derive both the **conditional mean remaining life**:

$$\mathrm{E}\{X \mid t, w_t\} = \sum_{\nu=0}^{\infty} R_t(x_\nu \mid w_t) \,, \quad (15)$$

and the **probability mass function** of the remaining lifetime X:

$$p(x_\nu \mid w_t) = \Pr\{X = \nu\, h \mid t, w_t\} = R_t(x_{\nu-1} \mid w_t) - R_t(x_\nu \mid w_t) \,, \quad \nu = 1, 2, \ldots \;. \quad (16)$$

A number of **functional forms** can be considered for $\varphi(t) \equiv \mathrm{E}\{W(t)\}$ and $\mathrm{V}\{W(t)\} \equiv \psi(t)$:

- the ***power-law*** function: $a\, t^b \ (a, b > 0)$
- the ***exponential*** function: $c\,[1 - \exp(-d\, t)] \ (c \cdot d > 0)$

It is immediate to verify that the **time-discrete Extended Gamma model includes**, as **special cases**:

(a) the **stationary Gamma process**, when mean and variance of $W(t)$ are both linear:

$$\mathrm{E}\{W(t)\} \equiv \varphi(t) = \theta_1\, t \qquad \text{and} \qquad \mathrm{V}\{W(t)\} \equiv \psi(t) = \theta_3\, t$$

(b) the **Gamma process with power-law** shape parameter $\eta(t) \propto t^b$, when mean and variance are both power-law functions with common shape parameter θ_0:

$$\varphi(t) = \theta_1\, t^{\theta_0} \qquad \text{and} \qquad \psi(t) = \theta_3\, t^{\theta_0}$$

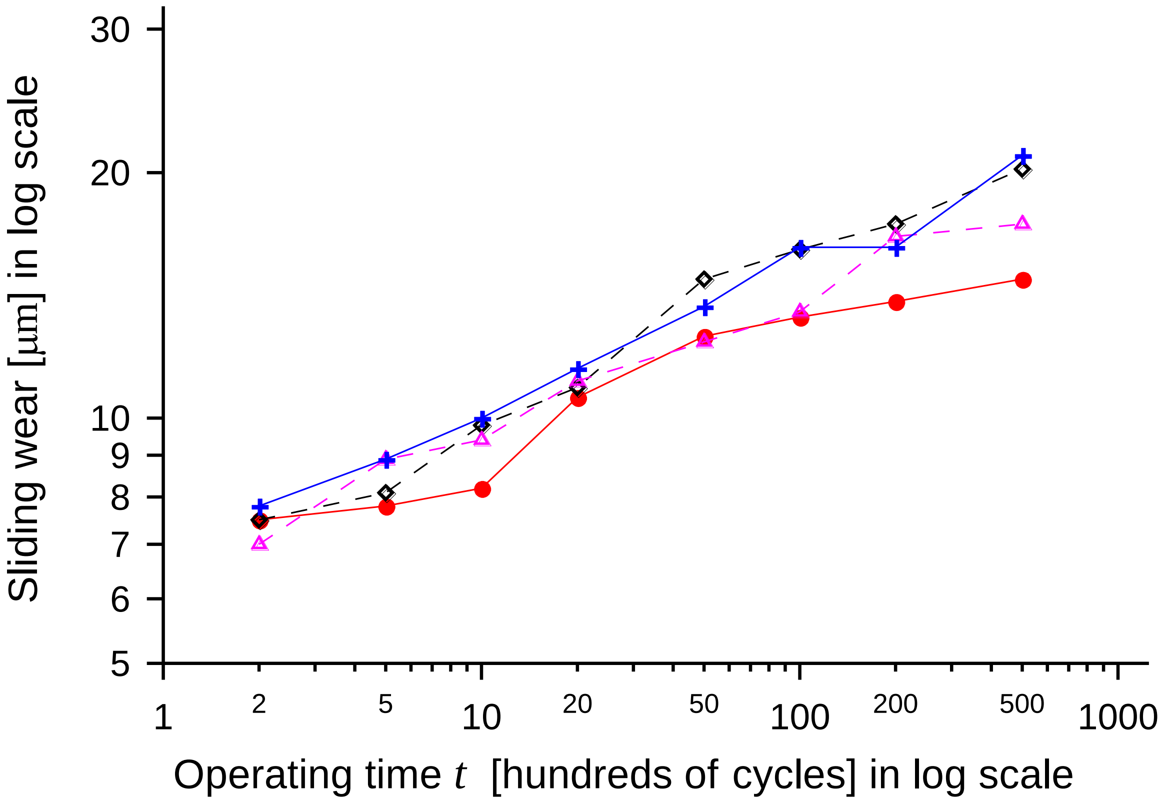
(c) the **Gamma process with exponential** shape parameter $\eta(t) \propto [1 - \exp(-bt)]$, when mean and variance are both exponential functions with common shape parameter θ_0:

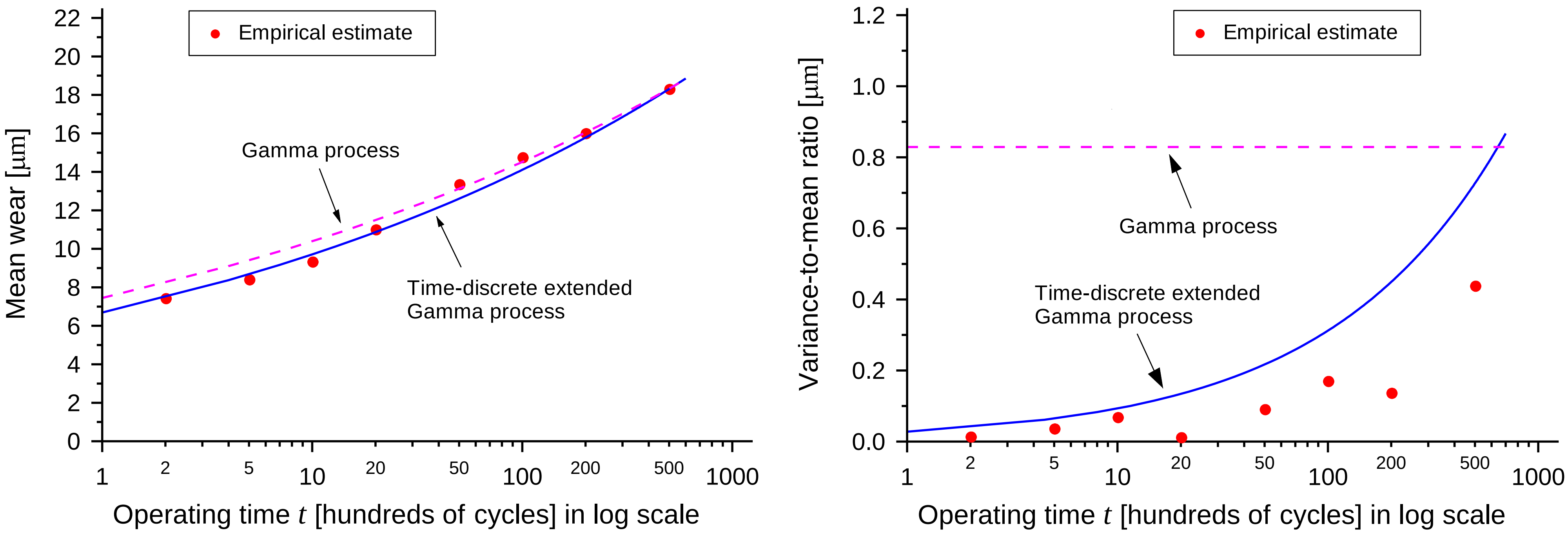
$$\varphi(t) = \theta_1\,[1 - \exp(-\theta_0\, t)] \qquad \text{and} \qquad \psi(t) = \theta_3\,[1 - \exp(-\theta_0\, t)]$$

In all these cases, the **variance-to-mean ratio of $W(t)$ is constant** with t and equal to θ_3 / θ_1.

Numerical application

Real dataset consisting of the **sliding wear data** (in microns) of four metal alloy specimens, measured at common selected ages up to 500 hundreds of cycles.





The time-discrete Extended Gamma process, with h=100 cycles and power-law function both for $\varphi(t)$ and $\psi(t)$ ($\hat{\theta}_1 = 6.69$, $\hat{\theta}_2 = 0.161$, $\hat{\theta}_3 = 0.183$, and $\hat{\theta}_4 = 0.687$), provides the better fit to the observed data.

The **likelihood ratio test rejects the null hypothesis** that $\theta_2 = \theta_4$, that is the hypothesis that the process is Gamma with power-law shape parameter $\eta(t)$: $\Lambda = 17.50$ and p-value = $2.9 \cdot 10^{-5}$.

The **property of "*independence of increments*"**, however, confines the use of Gamma processes to deterioration mechanisms where **the degradation growth** over the time interval $(t, t+\delta_T)$ **depends** (functionally) **on the current age** t and not on the (current) state of the unit.

In addition, if there is **empirical evidence that,** in absence of heterogeneity and/or measurement errors**, the variance** of the observed process **does not increase monotonically with the age** t, then **the Gamma process** (and the Extended Gamma process, as well as any other state-independent degradation process) **is not adequate to describe** the observed process because the property of **"*independence of increments*" forces the variance of the process to grow monotonically**.

Thus, we have to consider **a stochastic process able to describe degradation phenomena** where the **increments** (over disjoint time intervals) **are not independent** (**state-dependent processes**).

The **usual "*direct*" way** of modeling a deterioration phenomenon is in terms of the degradation level as a function of the operating time, say $\{W(t),\ t \geq 0\}$.

However, **we can also consider the "*inverse*" process,** that is the **time process** $\{T(w), w \geq 0\}$, *i.e.*, the process of the time T needed for reaching the degradation level w.

When the **"*direct*" process** $\{W(t),\ t \geq 0\}$ is a positive, increasing ***pure* state-dependent process** (that is, $\Delta W(t, \delta_T)$ depends physically only on the current state $W(t) = w_t$, and not on the current age t), then also the **"*inverse*" time process** $\{T(w), w \geq 0\}$ should be a positive and increasing process, **whose increments** $\Delta T(w_t, \delta_W) \equiv T(w_t + \delta_W) - T(w_t)$ over the degradation intervals $(w_t, w_t + \delta_W)$ **depend** physically **only on the current state** w_t of the unit, and not on its current age $t = T(w_t)$.

Thus, when the "*direct*" process $\{W(t),\ t \geq 0\}$ is a *pure* state-dependent process, independent on the age, the **time process** $\{T(w), w \geq 0\}$ **has "*independent time increments*",** and hence it **can be appropriately described** by the **Gamma process**.

Under the assumption that the "*inverse*" time process $\{T(w), w \geq 0\}$ is a (possibly) non-stationary Gamma process with shape parameter $\eta(w)$,

then **the time increment** $\Delta T(w_t, \delta_W)$ over the interval $(w_t, w_t + \delta_W)$ **follows a Gamma distribution** with shape parameter $\Delta\eta(w_t, \delta_W) = \eta(w_t + \delta_W) - \eta(w_t)$ and positive scale parameter β:

$$f_{\Delta T(w_t, \delta_W)}(\delta_T \mid w_t) = \frac{(\delta_T / \beta)^{\Delta\eta(w_t, \delta_W) - 1}}{\beta \; \Gamma[\Delta\eta(w_t, \delta_W)]} \exp(-\delta_T / \beta) \; , \quad \delta_T \geq 0 \; , \qquad (17)$$

- $\eta(w)$ is a non-negative, monotone increasing function of w, with $\eta(0) = 0$.

The notation $f_{\Delta T(w_t, \delta_W)}(\delta_T \mid w_t)$ allows us to stress the (functional) dependence on the current degradation level w_t

When $\eta(w)$ is a **linear function** of w, we have that $\Delta\eta(w_t, \delta_W) \propto w_t$ and hence the pdf of the time increment depends only on the length δ_W of the interval, and not on the current state $W(t) = w_t$.

The time process is then **homogeneous**.

The **mean and variance** of the "*inverse*" process $\{T(w), w \geq 0\}$ at the degradation level w are:

$$\mathrm{E}\{T(w)\} = \eta(w) \cdot \beta \qquad \text{and} \qquad \mathrm{V}\{T(w)\} = \eta(w) \cdot \beta^2 \; . \qquad (18)$$

Since the **remaining lifetime** X is the extra time the unit spends to exceed the threshold level $w_{\max}$, when starting from the current state w_t, the **probability distribution** of the remaining lifetime is **Gamma**, with shape parameter $\Delta\eta(w_t, w_{\max} - w_t) = \eta(w_{\max}) - \eta(w_t)$ and scale parameter β:

$$f_X(x|w_t) = \frac{(x/\beta)^{\Delta\eta(w_t, w_{\max} - w_t) - 1}}{\beta\ \Gamma[\Delta\eta(w_t, w_{\max} - w_t)]} \exp(-x/\beta) \quad . \tag{20}$$

The **mean remaining life** (MRL) is: $\mathrm{E}\{X | w_t\} = \beta\, \Delta\eta(w_t, w_{\max} - w_t)$.

The **residual reliability** of the unit is the probability that the remaining life X exceeds the time x, when starting from the current degradation level $W(t) = w_t \le w_{\max}$:

$$R_t(x|w_t) = \Pr\{X > x | w_t\} = 1 - \int_0^x \frac{(z/\beta)^{\Delta\eta(w_t, w_{\max} - w_t) - 1}}{\beta\ \Gamma[\Delta\eta(w_t, w_{\max} - w_t)]} \exp(-z/\beta)\, \mathrm{d}z \quad , \tag{21}$$

and the **reliability of a new unit** ($t = 0$ & $W(0) = 0$) is:

$$R(x) = \Pr\{X > x | 0\} = 1 - \int_0^x \frac{(z/\beta)^{\eta(w_{\max}) - 1}}{\beta\ \Gamma[\eta(w_{\max})]} \exp(-z/\beta)\, \mathrm{d}z \quad . \tag{22}$$

Since the process $\{T(w), w \geq 0\}$ is monotonically increasing, the **conditional probability distribution** that the **degradation growth** $\Delta W(t,\delta_T) = W(t+\delta_T) - W(t)$ over the time interval $(t,\ t+\delta_T)$ is less than δ_W, given the current state w_t, is:

$$F_{\Delta W(t,\delta_T)}(\delta_W \mid w_t) \equiv \Pr\{\Delta W(t,\delta_T) \leq \delta_W \mid w_t\} = \Pr\{\Delta T(w_t,\delta_W) > \delta_T \mid w_t\} \ . \qquad (23)$$

Thus, since $\{T(w), w \geq 0\}$ is a Gamma process, a degradation process $\{W(t), t \geq 0\}$ is said to be an **Inverse Gamma process** if the conditional Cdf of the degradation growth $\Delta W(t,\delta_T)$, given the current state, is given by:

$$F_{\Delta W(t,\delta_T)}(\delta_W \mid w_t) = \Pr\{\Delta T(w_t,\delta_W) > \delta_T \mid w_t\}$$

$$= 1 - \frac{\mathrm{IG}[\delta_T/\beta\,; \Delta\eta(w_t,\delta_W)]}{\Gamma[\Delta\eta(w_t,\delta_W)]} = 1 - \int_0^{\delta_T} \frac{(t/\beta)^{\Delta\eta(w_t,\delta_w)-1}}{\beta\ \Gamma[\Delta\eta(w_t,\delta_W)]} \exp(-t/\beta)\,\mathrm{d}t \ . \qquad (24)$$

It is evident that the (conditional) distribution of the degradation increment **depends on the current state** w_t, as well as on the time interval length δ_T, but **not on the current time** t.

Thus, the **Inverse Gamma process** can model **purely state-dependent degradation phenomena**.

If the shape function $\eta(w)$ is differentiable, the **conditional probability density function of** $\Delta W(t,\delta_T)$ can be obtained **by deriving** its Cdf with respect to δ_W:

$$f_{\Delta W(t,\delta_T)}(\delta_W \mid w_t) = \frac{\mathrm{d}F_{\Delta W(t,\delta_T)}(\delta_W \mid w_t)}{\mathrm{d}\delta_W} = -\frac{\mathrm{d}}{\mathrm{d}\delta_W}\int_0^{\delta_T} \frac{(t/\beta)^{\Delta\eta(w_t,\delta_W)-1}}{\beta\ \Gamma[\Delta\eta(w_t,\delta_W)]}\exp(-t/\beta)\,\mathrm{d}t\ . \qquad (25)$$

A **tractable expression** of the conditional pdf of $\Delta W(t,\delta_T)$ is given by:

$$f_{\Delta W(t,\delta_T)}(\delta_W \mid w_t) =$$

$$\frac{\eta'(w_t+\delta_W)}{\Gamma[\Delta\eta(w_t,\delta_W)]}\left\{\mathrm{IG}\left[\frac{\delta_T}{\beta};\Delta\eta(w_t,\delta_W)\right]\left[\psi[\Delta\eta(w_t,\delta_W)]-\ln(\delta_T/\beta)\right]+\sum_{k=0}^{\infty}\frac{(-1)^k(\delta_T/\beta)^{k+\Delta\eta(w_t,\delta_W)}}{[\Delta\eta(w_t,\delta_W)+k]^2\ k!}\right\} \qquad (26)$$

where $\eta'(a)=\mathrm{d}\eta(a)/\mathrm{d}a$ and $\psi(z)=\mathrm{d}\ln\Gamma(z)/\mathrm{d}z$ is the digamma function.

By setting $w_t=0$, we obtain the **density function of the degradation level** $W(t)$ reached at time t by a new unit:

$$f_{W(t)}(w|0)=\frac{\eta'(w)}{\Gamma[\eta(w)]}\left\{\left\{\mathrm{IG}\left[\frac{t}{\beta};\eta(w)\right]\left[\psi[\eta(w)]-\ln(t/\beta)\right]+\sum_{k=0}^{\infty}\frac{(-1)^k(t/\beta)^{k+\eta(w)}}{[\eta(w)+k]^2\ k!}\right\}\right. . \qquad (27)$$

The **conditional mean** $\mathrm{E}\{\Delta W(t,\delta_T)|w_t\}$ of the degradation growth over the time interval $(t, t+\delta_T)$ of length δ_T, given the current state w_t, follows:

$$\mathrm{E}\{\Delta W(t,\delta_T)|w_t\}=\int_0^\infty [1-F_{\Delta W(t,\delta_T)}(\delta_W|w_t)]\,\mathrm{d}\delta_W=\int_0^\infty \frac{\mathrm{IG}[\delta_T/\beta;\Delta\eta(w_t,\delta_W)]}{\Gamma[\Delta\eta(w_t,\delta_W)]}\,\mathrm{d}\delta_W\ , \tag{28}$$

and the **conditional variance** $\mathrm{V}\{\Delta W(t,\delta_T)|w_t\}$ is given by:

$$\mathrm{V}\{\Delta W(t,\delta_T)|w_t\}=\int_0^\infty (\delta_W-E\{\Delta W(t,\delta_T)|w_t\})^2\, f_{\Delta W(t,\delta_T)}(\delta_W|w_t)\,\mathrm{d}\delta_W\ . \tag{29}$$

It can be show that the **variance** $\mathrm{V}\{W(t)\}$ of the degradation process **is not constrained to increase monotonically** with t, because the **IG process has not independent increments**.

For **a new unit**, the mean degradation level at the time t is:

$$\mathrm{E}\{W(t)|0\} = \int_0^\infty \frac{\mathrm{IG}[t/\beta; \eta(\delta_W)]}{\Gamma[\eta(\delta_W)]} \, \mathrm{d}\delta_W \quad . \tag{30}$$

The **second derivative** w.r.t. t is given by:

$$\frac{\mathrm{d}^2 \mathrm{E}\{W(t)|0\}}{\mathrm{d}t^2} = \exp(-t/\beta) \int_0^\infty \frac{(t/\beta)^{\eta(w)-1}}{\beta \; \Gamma[\eta(w)]} \left(\frac{\eta(w)-1}{t} - \frac{1}{\beta} \right) \mathrm{d}w \quad , \tag{31}$$

and is negative for small t values, even if $\eta(w) \propto w$.

Thus, **even if the time process** $\{T(w), w \geq 0\}$ **is homogeneous** (so that $\mathrm{E}\{T(w)\}$ increases linearly), **the degradation mean** $\mathrm{E}\{W(t)|0\}$ **does not increase linearly.**

⇩

In particular, when $\eta(w) \propto w$, **the** $\mathrm{E}\{W(t)|0\}$ **curve is concave** (downwards), especially for small t values, and tends to be linear as t increases.

Several **functional forms** for the $\eta(w)$ function are considered, which determine the behavior of the mean and variance of the time process $\{T(w), w \geq 0\}$:

$$\mathrm{E}\{T(w)\} = \eta(w) \cdot \beta \qquad \text{and} \qquad \mathrm{V}\{T(w)\} = \eta(w) \cdot \beta^2 , \tag{32}$$

and consequently the behavior of the Inverse Gamma degradation process $\{W(t), t \geq 0\}$:

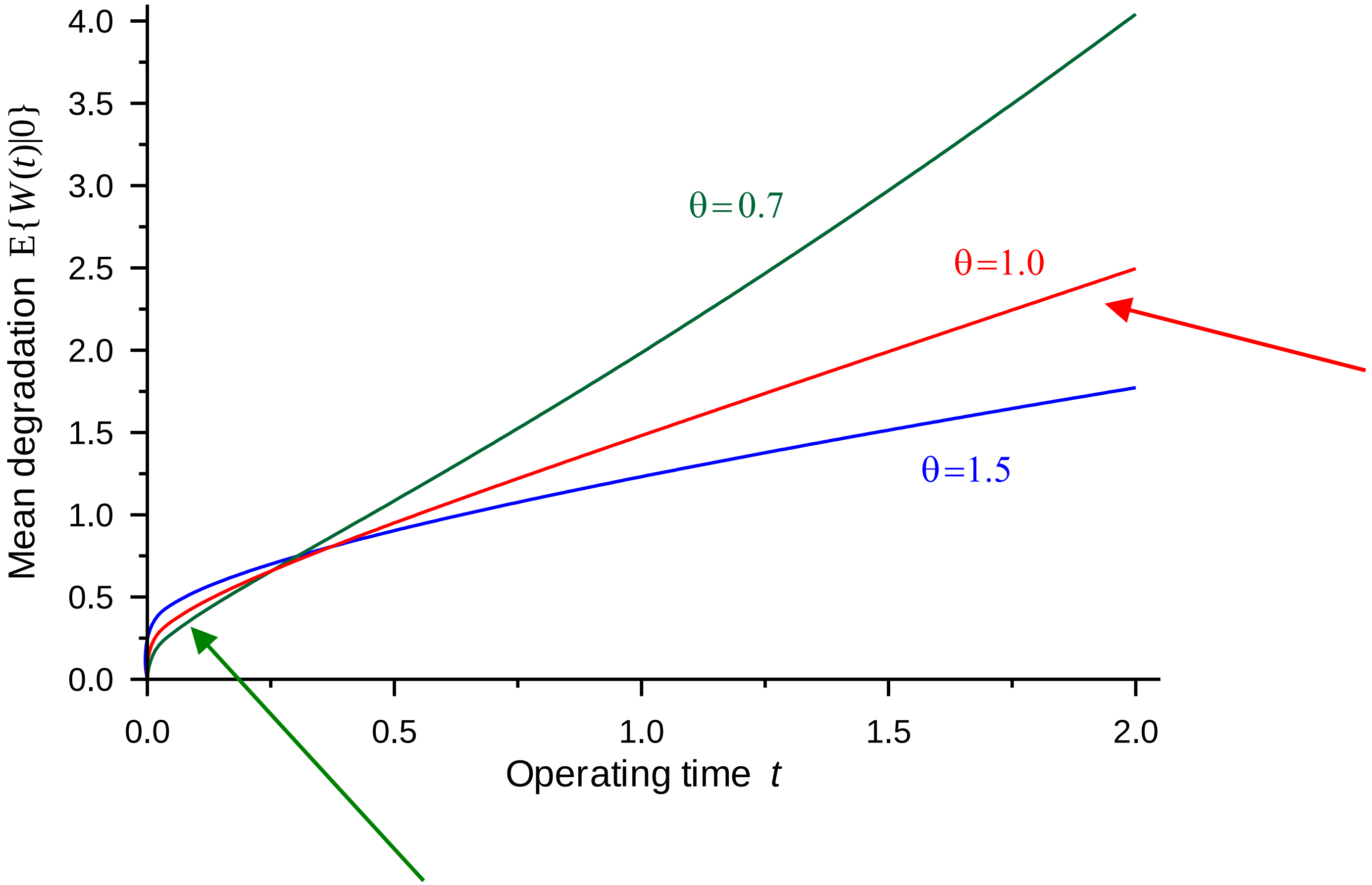
a) the ***exponential*** (Exp) function: $\eta(w) = \alpha\,[\exp(w/\theta) - 1]$, with $-\infty < \alpha, \theta < \infty$, and $\alpha \cdot \theta > 0$.

b) the ***power-law*** (PL) function: $\eta(w) = (w/\alpha)^{\theta}$, with $\alpha, \theta > 0$.

c) the ***logarithmic*** (Log) function: $\eta(w) = \alpha\,\theta \ln(1 + w/\theta)$, with $\alpha > 0$.

d) the ***linear-exponential*** (LE) function: $\eta(w) = \alpha\{w + \theta[1 - \exp(-w/\theta)]\}$, with $\alpha > 0$.

Mean degradation curve of the Inverse Gamma processes with *power-law* $\eta(w) = (w/\alpha)^{\theta}$.



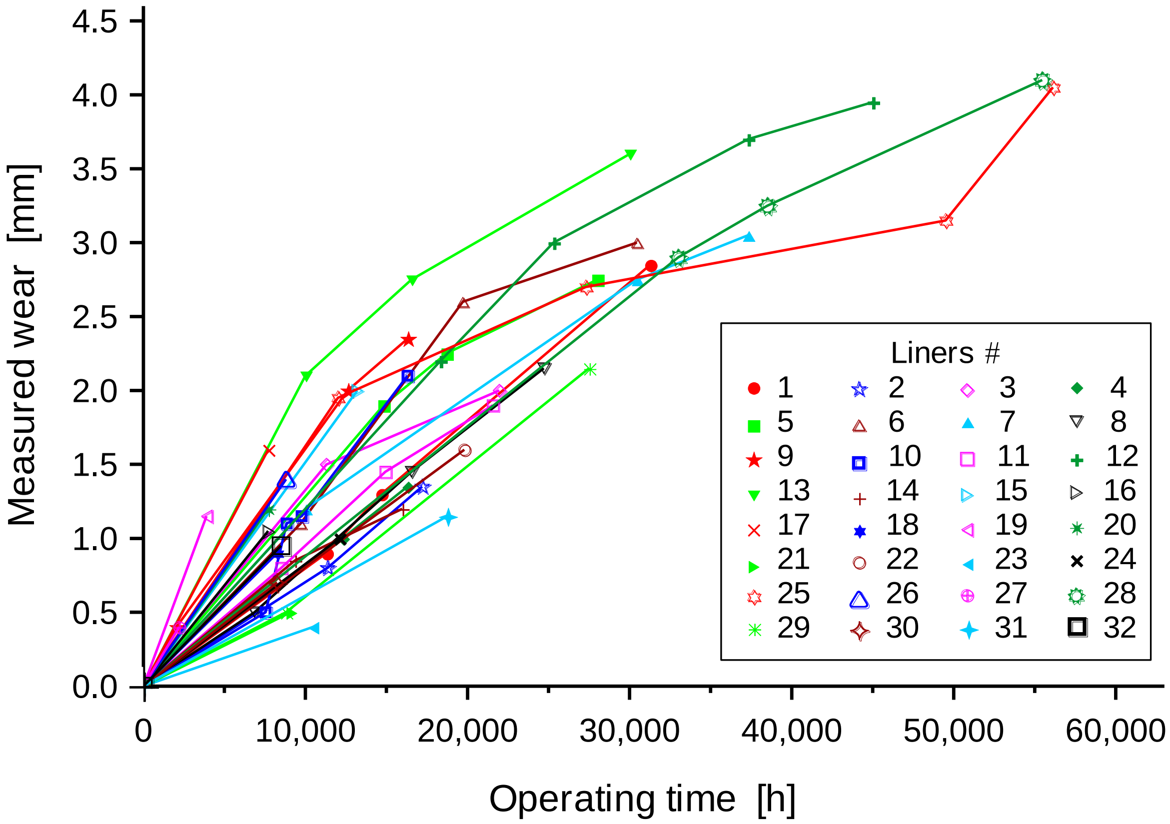
Even if **the time process** $\{T(w), w \geq 0\}$ **is homogeneous** (that is, when $\theta = 1$), **the degradation mean** $\mathrm{E}\{W(t)|0\}$ **does not increase exactly linearly** with t ,but is concave (downwards)**.**

Even when $\theta \leq 1$, the $\mathrm{E}\{W(t)|0\}$ curves are **concave** (downwards) for small t values, and then becomes convex**.**

Numerical application

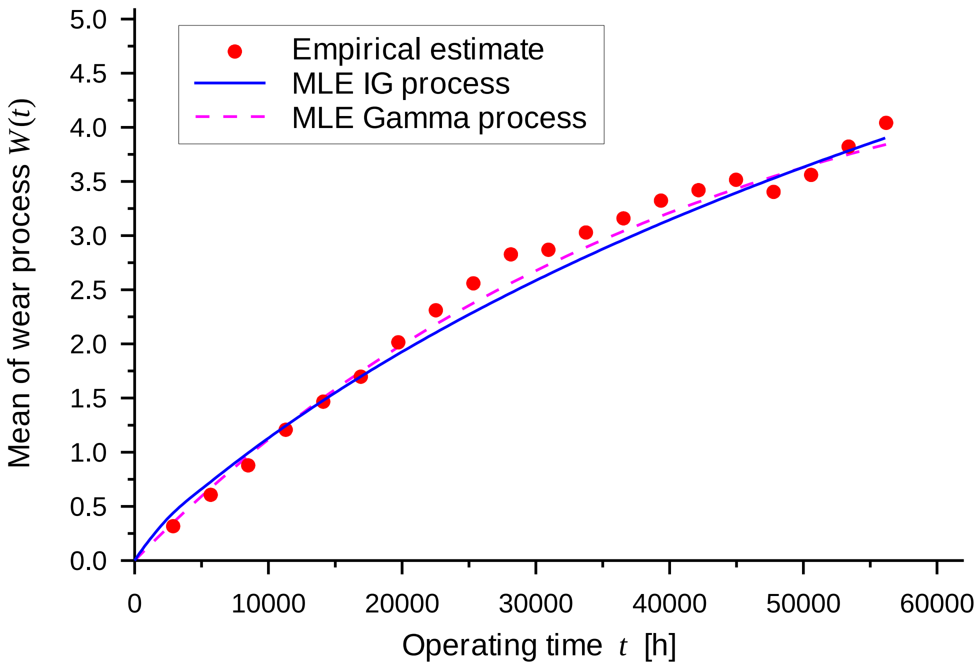
Real dataset consisting in the **wear process of 32 cylinder liners** of the Diesel engines equipping some identical cargo ships of the Grimaldi Lines.

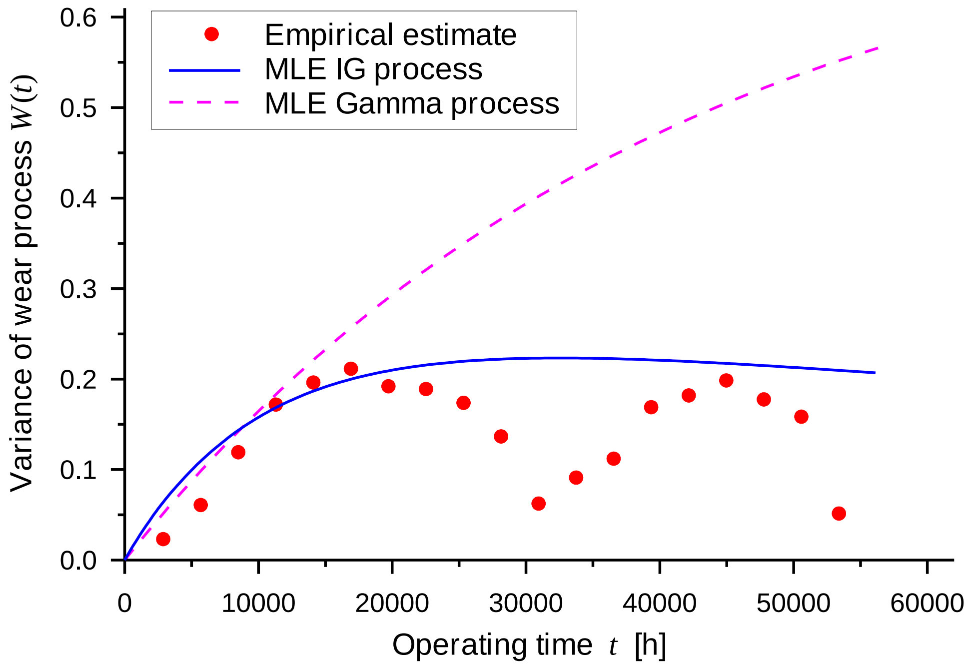
The data were collected from January 1999 to August 2006.



The observation period greatly differs from liner to liner, ranging from a minimum of 2,200 hours up to a maximum of 56,120 hours.

The number of inspections performed on each liner varies from 1 up to 4.





——— Inverse Gamma process on the wear process $\{W(t), t \geq 0\}$ with $\eta(w) = \alpha\,[\exp(w/\theta) - 1]$

$(\hat{\alpha} = 11.19,\ \hat{\theta} = 2496.5$ hours, $\hat{\beta} = 3.468$ mm, $\hat{\ell} = 43.30,\ \text{AIC} = -80.60)$

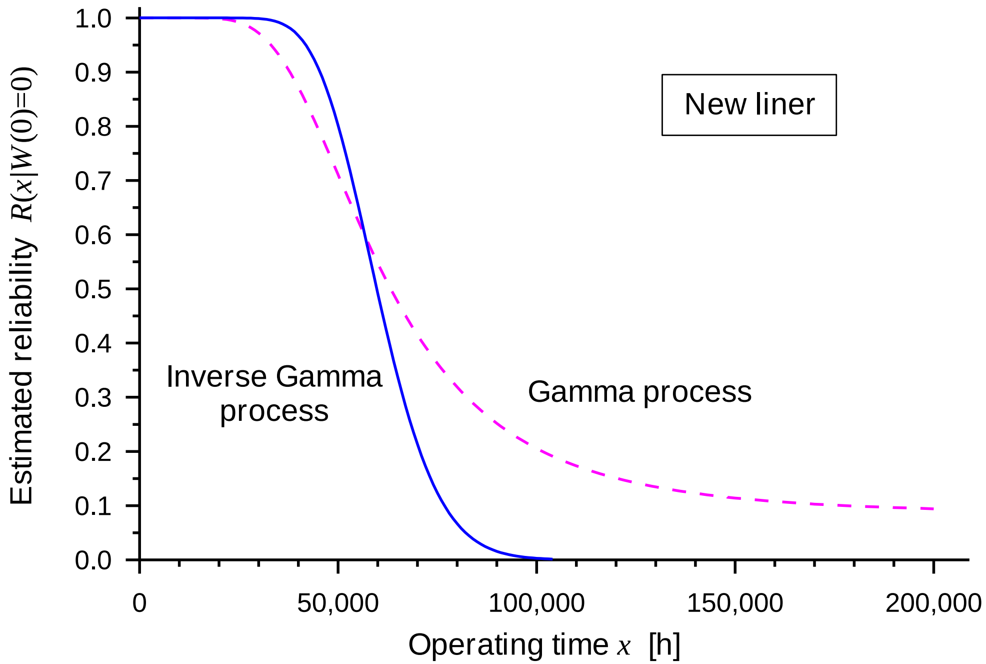
- - - - Gamma process on the wear process $\{W(t), t \geq 0\}$ with $\eta(t) = \alpha\,[\exp(t/\theta) - 1]$

$(\hat{\alpha} = -35.72,\ \hat{\theta} = -4.076 \cdot 10^4$ hours, $\hat{\beta} = 0.1471$ mm, $\hat{\ell} = 22.92,\ \text{AIC} = -39.84)$

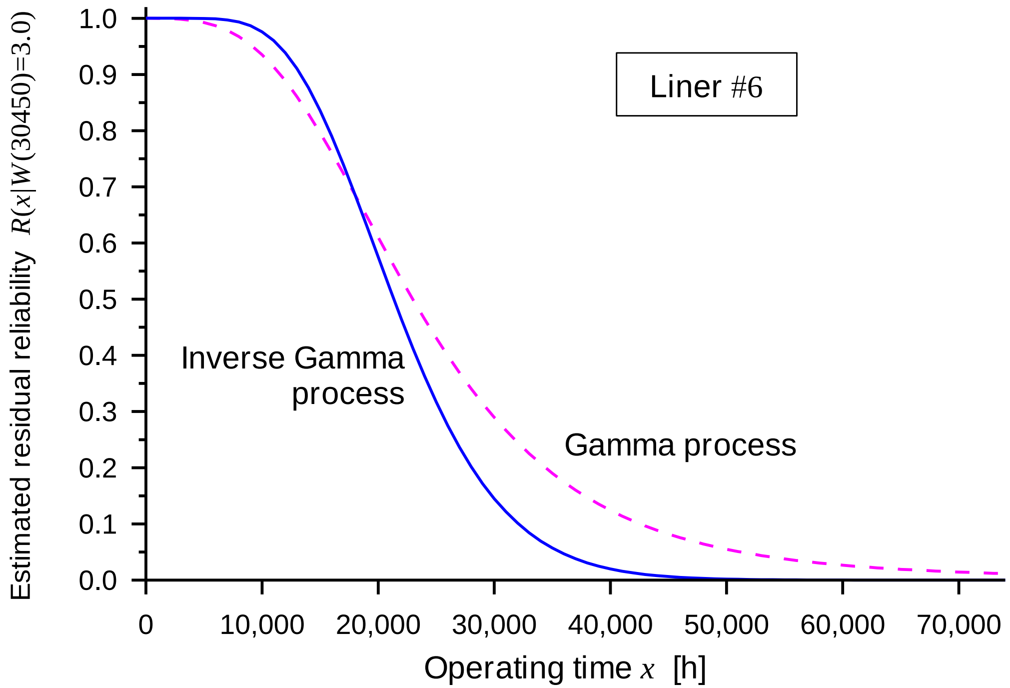
The above $\eta(w)$ & $\eta(t)$ functions provide the better fit to the data in terms of the Akaike Information Criterion (AIC).

The **IG process has to be preferred to the Gamma process** because the AIC value under the IG process $(\text{AIC} = -80.60)$ is much smaller than the AIC value under the Gamma process $(\text{AIC} = -39.84)$.

Reliability of a new liner



Residual reliability of liner #6



—— Inverse Gamma process on the wear process $\{W(t), t \geq 0\}$ with $\eta(w) = \alpha[\exp(w/\theta) - 1]$

- - - - Gamma process on the wear process $\{W(t), t \geq 0\}$ with $\eta(t) = \alpha[\exp(t/\theta) - 1]$.

The Gamma process **underestimates** the reliability and the residual reliability at small operating times and greatly **overestimates** the reliability at large *x*.

In some cases, however, the observed degradation phenomena can have **increments** $\Delta W(t,\delta_T)$ **which depend both on the current age** t **and on the current state** $W(t)=w_t$. In these cases, therefore, neither the Gamma process, nor the Inverse Gamma process, can adequately describe the observed process.

A suitable model is the **Transformed Gamma (TG) process**.

An increasing, continuous-time, degradation process $\{W(t);t\geq 0\}$ is said to be a TG processes if **the (conditional) pdf of** $\Delta W(t,t+\Delta t)$ is given by

$$f_{\Delta W(t,\delta_T)}(w|t,w_t)= g'(w_t+w)\frac{[\Delta g(w_t,w)]^{\Delta\eta(t,\delta_T)-1}}{\Gamma[\Delta\eta(t,\delta_T)]}\exp[-\Delta g(w_t,w)] \qquad (33)$$

- $\eta(t)$ is a non-negative, monotone increasing function of t ("*age function*"), with $\eta(0)=0$,
- $g(w)$ is a non-negative, monotone increasing and differentiable function of w ("*state function*"), with $g(0)=0$,
- $g'(w_t+w)$ is the derivative of $g(w)$ evaluated at w_t+w,
- $\Delta g(w_t,w)\equiv g(w_t+w)-g(w_t)$ & $\Delta\eta(t,\delta_T)=\eta(t+\delta_T)-\eta(t)$.

It is evident that, if the age and state functions are not linear, the **degradation increment** $\Delta W(t,\delta_T)$ **depends both on the current age** t **and the current state** w_t, in addition to the interval length δ_T.

The TG process **includes as special case the Gamma process**.

1) If **the state function is linear**, say $g(w)=w/\alpha$, the TG process reduces to the *pure* age-dependent **Gamma process**, where the distribution of the degradation increment no more depends on the current state w_t but only (at most) on the current age t.

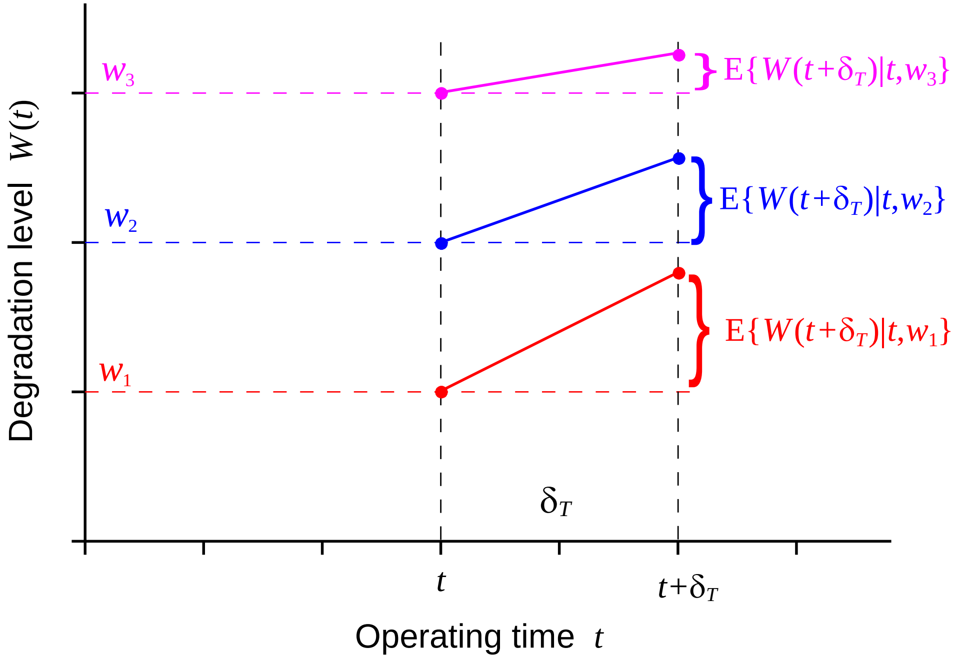
2) If **both** $g(w)$ **and** $\eta(t)$ **are linear**, the TG process reduces to a Gamma process with stationary increments.

Finally, if **the age function is linear**, say $\eta(t)=t/a$,
the TG process reduces to a ***pure*** **state-dependent (SD) process**
(*i.e.*, with increments that, given the current state w_t, are independent of the current age *t*).

The **conditional mean** of the degradation increment $\Delta W(t,\delta_T)$ over the interval $(t, t+\delta_T)$, given the current age t and degradation level $W(t)=w_t$, is:

$$\mathrm{E}\{\Delta W(t,\delta_T)\,|\,t,w_t\} = \int_0^\infty w\, f_{\Delta W(t,\delta_T)}(w\,|\,t,w_t)\,\mathrm{d}w \quad . \tag{34}$$

The behavior of the state function $g(w)$ **has a clear effect** on the conditional mean of the degradation increment $\Delta W(t,\delta_T)$**.**



For example, **if** $g(w)$ **is convex** (downward)**,** the conditional mean of the increment $\Delta W(t,\delta_T)$ at the time t, given that $W(t)=w_t$, is **a decreasing function** of w_t.

The conditional **Cdf of the degradation increment** $\Delta W(t,\delta_T)$, given the current age t and state w_t, is

$$F_{\Delta W(t,\delta_T)}(w|t,w_t) = \frac{\mathrm{IG}[\Delta g(w_t,w);\Delta\eta(t,\delta_T)]}{\Gamma[\Delta\eta(t,\delta_T)]} \qquad (35)$$

where $\mathrm{IG}(x;a) = \int_0^x z^{a-1}\exp(-z)\,\mathrm{d}z$ is the incomplete Gamma function.

Under the TG process, **the conditional residual reliability** is given by:

$$R_t(x|t,w_t) = \Pr\{X > x|t,w_t\} = \Pr\{\Delta W(t,x) \le w_{\max} - w_t\,|t,w_t\}$$

$$= \frac{\mathrm{IG}[\Delta g(w_t,w_{\max}-w_t);\Delta\eta(t,x)]}{\Gamma[\Delta\eta(t,x)]} = \int_0^{\Delta g(w_t,w_{\max}-w_t)} \frac{z^{\Delta\eta(t,x)-1}}{\Gamma[\Delta\eta(t,x)]}\exp(-z)\,\mathrm{d}z \; . \qquad (36)$$

and hence it **depends both on** t **and on** w_t (not only through $w_{\max} - w_t$).

Clearly, the **reliability of a new unit** is given by:

$$R(x|0,0) = \frac{\mathrm{IG}[g(w_{\max});\eta(x)]}{\Gamma[\eta(x)]} \; . \qquad (37)$$

If the age function $\eta(t)$ is differentiable, we can obtain the **conditional pdf of the remaining lifetime** X by deriving the reliability function with respect to x:

$$f_X(x|t,w_t) = -\frac{1}{\mathrm{d}x} \int_0^{\Delta g(w_t, w_{\max}-w_t)} \frac{u^{\Delta\eta(t,x)-1}}{\Gamma[\Delta\eta(t,x)]} \exp(-u)\,\mathrm{d}u$$

$$= \frac{\Delta\eta'(t,x)}{\Gamma[\Delta\eta(t,x)]} \Big\{ \mathrm{IG}\big[\Delta g(w_t, w_{\max}-w_t); \Delta\eta(t,x)\big] \times \big(\psi[\Delta\eta(t,x)] - \ln[\Delta g(w_t, w_{\max}-w_t)]\big) \cdot$$

$$+ \frac{[\Delta g(w_t, w_{\max}-w_t)]^{\Delta\eta(t,x)}}{[\Delta\eta(t,x)]^2} \, {}_2F_2\big(\Delta\eta(t,x), \Delta\eta(t,x); \Delta\eta(t,x)+1, \Delta\eta(t,x)+1; -\Delta g(w_t, w_{\max}-w_t)\big) \Big\} \quad (38)$$

- $\psi(\bullet)$ is the digamma function,
- $\Delta\eta'(t,x) \equiv \mathrm{d}\Delta\eta(t,x)/\mathrm{d}x = \eta'(t+x)$,
- ${}_2F_2(\bullet)$ is the generalized hypergeometric function of order $(2,2)$.

Since the generalized hypergeometric function ${}_2F_2(\bullet)$ of order $(2,2)$ can be evaluated by:

$$ {}_2F_2(\bullet) = \sum_{k=0}^{\infty} \frac{(\Delta\eta(t,x))^2 \, [-\Delta g(w_t, w_{\max} - w_t)]^k}{(\Delta\eta(t,x)+k)^2 \, k!} \tag{39} $$

the **conditional pdf of the remaining lifetime**, given the current age t and state w_t, is given by:

$$ f_X(x|t,w_t) = \frac{\eta'(t+x)}{\Gamma[\Delta\eta(t,x)]} \Big\{ \mathrm{IG}[\Delta g(w_t, w_{\max} - w_t); \Delta\eta(t,x)] \times \big(\psi[\Delta\eta(t,x)] - \ln[\Delta g(w_t, w_{\max} - w_t)]\big) + \sum_{k=0}^{\infty} \frac{(-1)^k \, [\Delta g(w_t, w_{\max} - w_t)]^{\Delta\eta(t,x)+k}}{(\Delta\eta(t,x)+k)^2 \, k!} \Big\} \,. \tag{40} $$

Note that this sum converged quite quickly in our applications

The **mean remaining lifetime** can be obtained by numerical integration of the residual reliability

$$ \mathrm{E}\{X|t,w_t\} = \int_0^{\infty} R_t(x|t,w_t)\,\mathrm{d}x = \int_0^{\infty} \frac{\mathrm{IG}[\Delta g(w_t, w_{\max} - w_t); \Delta\eta(t,x)]}{\Gamma[\Delta\eta(t,x)]}\,\mathrm{d}x \,. \tag{41} $$

In order to obtain a fully operative formulation of the TG class of processes, it is necessary to give a **functional form** to the state function $g(w)$ and to the age function $\eta(t)$.

Possible choices both for the state and the age function are the following:

	Power-law	Exponential
State function	$g(w)=(w/\alpha)^{\beta}$, $\alpha,\beta>0$	$g(w)=\alpha\,[\exp(w/\beta)-1]$, $-\infty<\alpha,\beta<\infty$, $\alpha\cdot\beta>0$
Age function	$\eta(t)=(t/a)^{b}$, $a,b>0$	$\eta(t)=a\,[\exp(t/b)-1]$, $-\infty<a,b<\infty$, $a\cdot b>0$

Both these functional forms can reduce (or can tend) to a linear function. In particular:

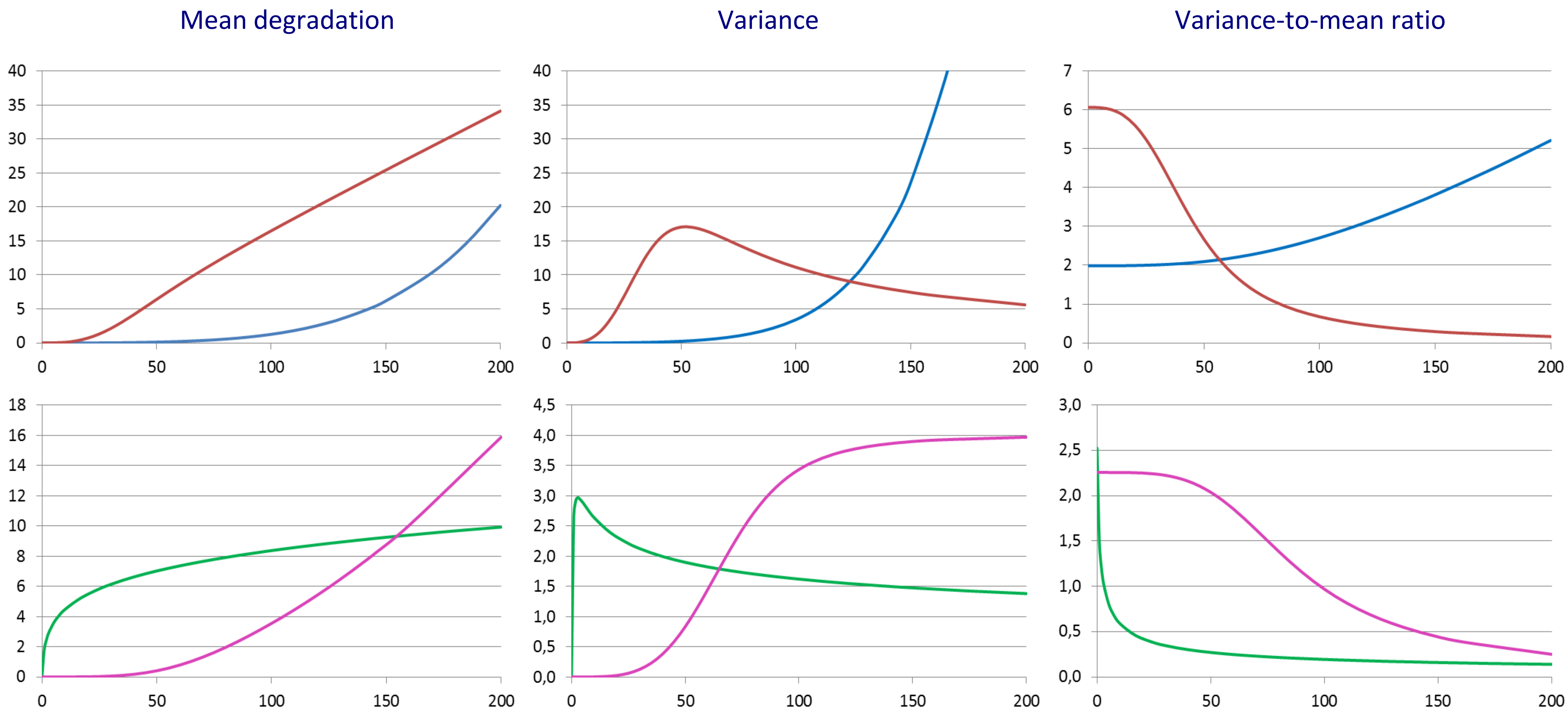
- the ***power-law*** function $g(w)=(w/\alpha)^{\beta}$ is linear if $\beta=1$,
- the ***exponential*** function $g(w)=(\beta/\alpha)[\exp(w/\beta)-1]$ is linear for $|\beta|\to\infty$,

thus **allowing the TG model to reduce** to a pure age-dependent (when $g(w)$ is linear), or a pure state-dependent (when $\eta(t)$ is linear), or finally to a stationary process (both $g(w)$ and $\eta(t)$ linear).

If $g(w)=(w/\alpha)^{\beta}$, the **mean degradation level** and the **variance** of a new unit are in closed form, regardless of the functional form of $\eta(t)$:

$$\mathrm{E}\{W(t)|0,0\}=\alpha\,\frac{\Gamma[\eta(t)+1/\beta]}{\Gamma[\eta(t)]} \tag{42}$$

$$\mathrm{V}\{W(t)|0,0\}=\frac{\alpha^{2}}{\Gamma[\eta(t)]}\left(\Gamma[\eta(t)+2/\beta]-\frac{\Gamma[\eta(t)+1/\beta]^{2}}{\Gamma[\eta(t)]}\right)\;. \tag{43}$$

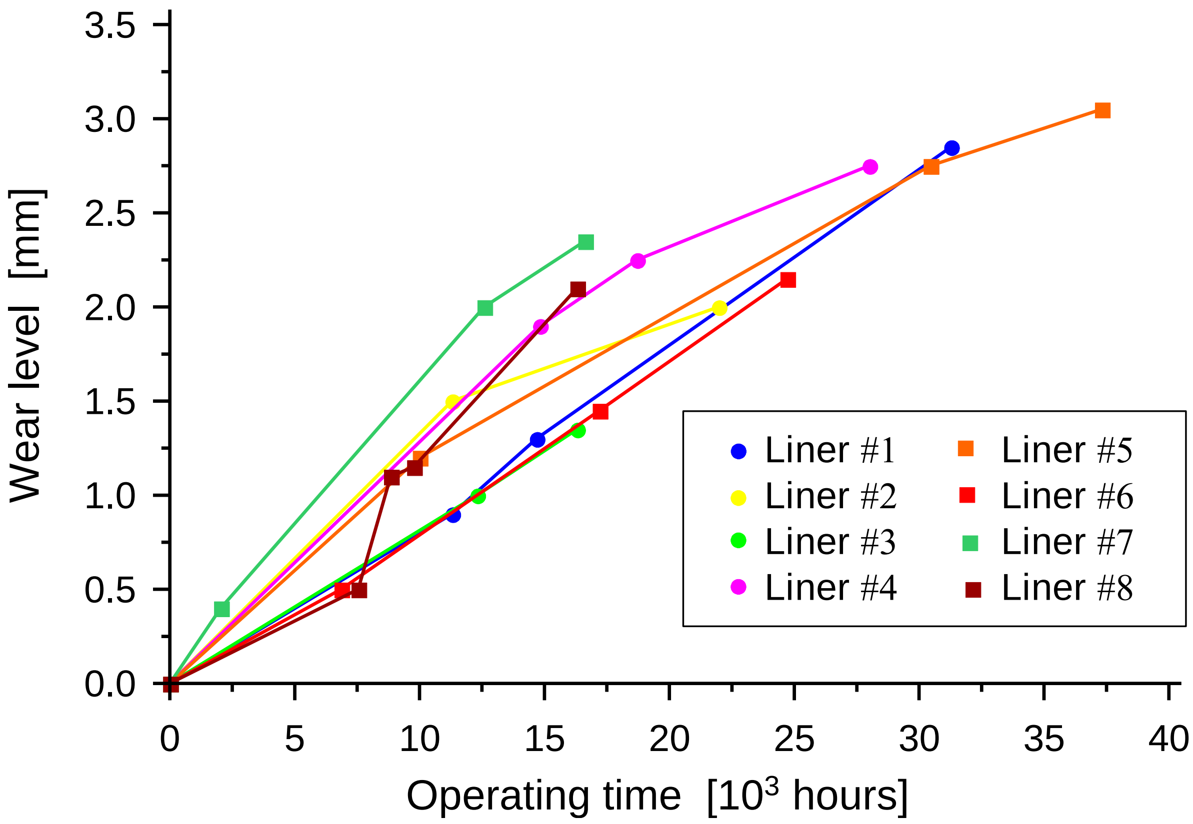


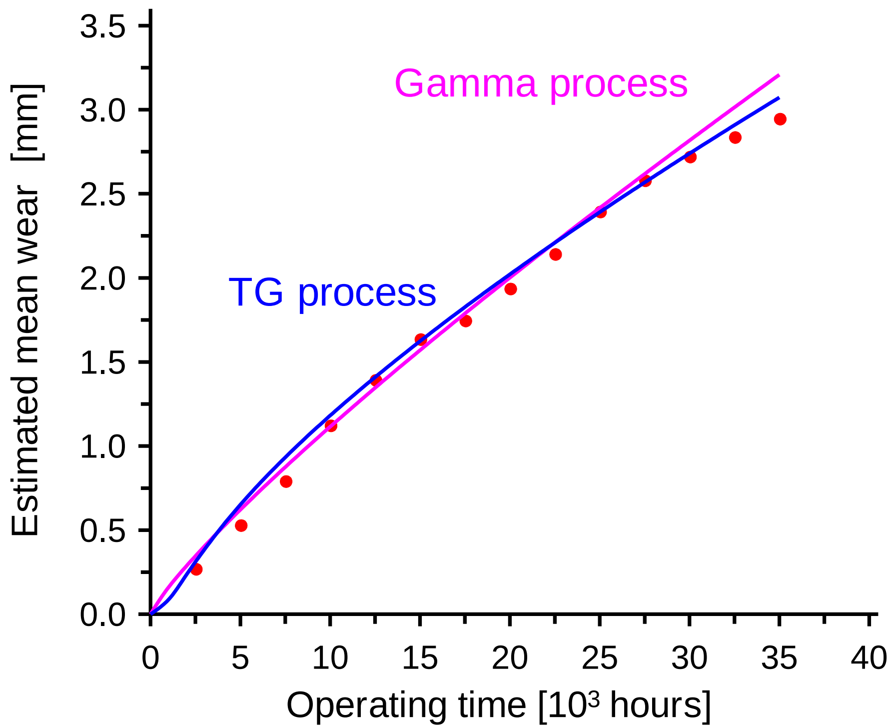
The behaviour of the mean, the variance, and the variance-to-mean ratio of the TG process with power-law $\eta(t)$ and $g(w)$ functions, for selected b and β values: $b=3, \beta=0.7$, $b=3, \beta=3$, $b=0.7, \beta=3$, and $b=4, \beta=2$.

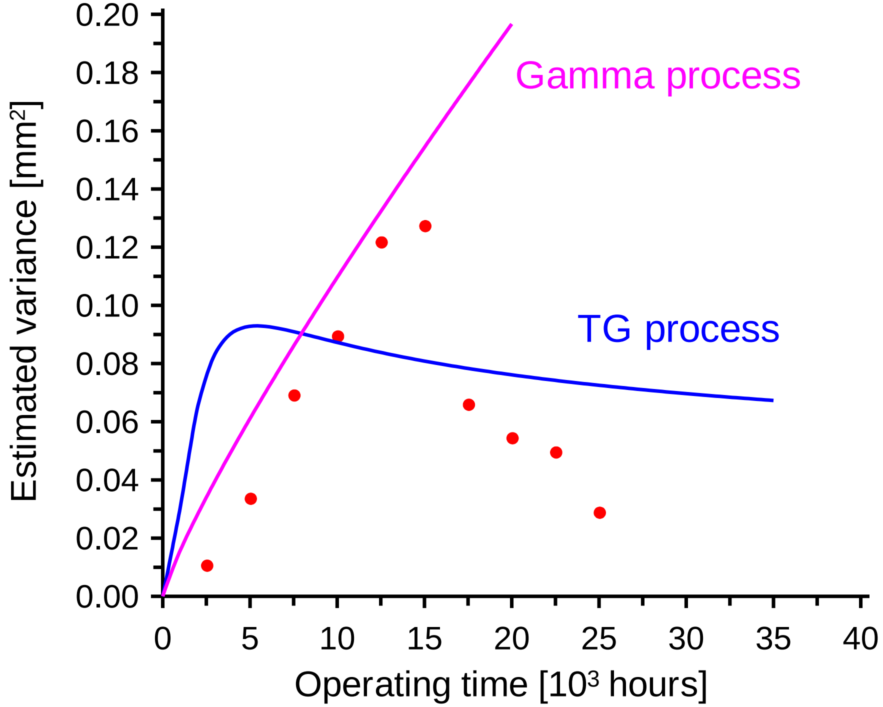
Large flexibility of the TG process in describing mean and variance behaviours.

Numerical application

Let us consider the **wear data of the 8 cylinder liners**
of the Diesel engine equipping a cargo ship of the Grimaldi Lines.







TG process with power-law functions $\eta(t)=(t/a)^b$ and $g(w)=(w/\alpha)^\beta$

$(\hat{a}=5107$ hours, $\hat{b}=1.701$, $\hat{\alpha}=0.750$ mm, $\hat{\beta}=2.313$, $\hat{\ell}=0.59)$.

Gamma process with power-law functions $\eta(t)=(t/a)^b$

$(\hat{a}=562$ hours, $\hat{b}=0.844$, $\hat{\alpha}=0.0983$ mm, $\hat{\ell}=-2.44)$.

The **likelihood ratio test suggests to reject the null hypothesis** that the process is **Gamma**:

$\Lambda=6.06$ and $p-\text{value}=0.014$.

The threshold limit is assumed to be $w_{max} = 4$ mm (the warranty limit).

The exceeding of the threshold limit does not generally produce a sudden break down of the engine, because it continues to operate even if the threshold limit is exceeded, but the "*failure*" occurrence reduces the performance of the engine and increases the consumption of fuel and motor oil (resulting in increased toxic emissions).

The unit is inspected at time t (age/operating time of the liner).

The **liner maintenance is optimized** based on the planned inspections.
In particular, at each inspection:

- if the current wear of the liner exceeds the threshold limit $w_{max} = 4$ mm, the liner **is immediately replaced with a new one**,
- otherwise, we can:

  a) **defer the substitution** of the liner to the time $t + \tau$ of the next inspection, or

  b) **replace now** the used liner with a new one.

The **utility of a used liner** at the future time $t+\tau$ of the next inspection depends on the unknown degradation level $w_{t+\tau}$ and is:

Total cost of new liner (including the cost of mounting)

Conditional mean remaining life at the time $t+\tau$

Mean life of a new liner

$$U(w_{t+\tau};\tau)=\begin{cases} c_L \; \mathrm{E}\{X \mid t+\tau, w_{t+\tau}\}/\mathrm{E}\{T\} & w_{t+\tau} < w_{\max} \\ 0 & w_{t+\tau} = w_{\max} \\ -[(w_{t+\tau} - w_{\max})/c]^d & w_{t+\tau} > w_{\max} \end{cases}$$

Positive utility (liner not failed)

Null utility (threshold value reached)

Negative utility (liner failed)

Parameters of cost due to failure

Likewise, the **utility of a new liner** at the time τ of the next inspection is:

$$U(w_{\tau};\tau)=\begin{cases} c_L \; \mathrm{E}\{X \mid \tau, w_{\tau}\}/\mathrm{E}\{T\} & w_{\tau} < w_{\max} \\ 0 & w_{\tau} = w_{\max} \\ -[(w_{\tau} - w_{\max})/c]^d & w_{\tau} > w_{\max} \end{cases}$$

The **expected utility loss** over the interval $(t, t+\tau)$ up to the next inspection **associated to the action *a*)**: *deferring the substitution of the liner to the time of the next inspection*, is

$$E\{UL_a(\tau)\} = \left(U(w_t) - \int_{w_t}^{w_{max}} U(w;\tau) \cdot f_{W(t+\tau)}(w|t,w_t)\,\mathrm{d}w \right) - \int_{w_{max}}^{\infty} U(w;\tau) \cdot f_{W(t+\tau)}(w|t,w_t)\,\mathrm{d}w$$

Expected reduction of the liner value
($U(w_t) = c_L E\{X|t,w_t\}/E\{T\}$ is the current utility of the liner)

Expected cost due to failure of used liner
(including the minus sign)

The **expected utility loss** over the interval up to the next inspection **associated to the action *b*)**: *replacing now the used liner with a new one*, is

$$E\{UL_b(\tau)\} = \left(U(0) - \int_{0}^{w_{max}} U(w;\tau) \cdot f_{W(\tau)}(w)\,\mathrm{d}w \right) - \int_{w_{max}}^{\infty} U(w;\tau) \cdot f_{W(\tau)}(w)\,\mathrm{d}w + U(w_t)$$

Expected reduction of the value of the new liner
($U(0) = c_L$ is the utility of the new liner)

Residual value of the used (substituted) liner

Expected cost due to failure of new liner
(including the minus sign)

By assuming: $c_L = 10{,}000$ €, $c = 6.07 \cdot 10^{-5}$ mm & $d = 1.2$, the **differences Δ_i between the expected utility losses** associated to actions *a*) *deferring the substitution of the liner to a later time*, and *b*) *replacing now the liner with a new one*, **when the next inspection is planned after $\tau = 10{,}000$ hours**, under the TG process and the Gamma process are:

	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$	$i = 6$	$i = 7$	$i = 8$
Age t_i [h]	31,270	21,970	16,300	28,000	37,310	24,710	16,620	16,300
Wear w_i [mm]	2.85	2.00	1.35	2.75	3.05	2.15	2.35	2.10
Δ_i under TG process [€]	3726	5883	7136	4144	**2819**	5441	5850	6292
Δ_i under Gamma process [€]	1933	5713	7447	2739	-896	5329	4428	5233

A positive difference Δ_i implies that the substitution of the liner *i* should be deferred

The proposed maintenance strategy leads to **defer the replacement of all the liners** under the TG model. On the contrary, under the (inadequate) Gamma process, the **liner 5 should be substituted now**, thus producing a **unnecessary cost of 2819 €**.

For $\tau = 15{,}000$ hours, only liner #5 has to be substituted under the TG process ($\Delta_5 = -2951$€), whereas under the Gamma process we should replace liners #1, 4 & 5.

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Thank you for your attention!

Merci beaucoup pour votre attention!