Sensitivity analysis for the block replacement policy

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AMMSI, 20 January 2015

Maintenance and sensitivity analysis

- Given a parametric distribution of time-to-failure (or lifetime), maintenance cost and parameters can be optimized
- When the parameters of lifetime distribution are unknown, estimation is required
- Optimal maintenance cost and parameters depend on the parameters estimation
- Estimation based on a set of time-to-failure observations
- Optimality sensitive to the estimation results
- Sensitivity analysis: necessary for an efficient maintenance planning

Sensitivity analysis & Maintenance

Block replacement policy

- First step to propose analytical expressions for the sensitivity analysis
- Focus on the sensitivity analysis of block replacement policy
- The optimisation of this policy requires the optimization of only one parameter.

Block replacement policy context

- Let T be the time-to-failure (or lifetime) of the device.
- The distribution of T depends on some parameter $\theta \in \Theta \subset \mathbb{R}^p$
- Usual notations: $f_T(\cdot;\theta)$ the probability distribution function, $F_T(\cdot;\theta)$ the cumulative distribution function, and $S_T(\cdot;\theta)$ the survival function (or reliability).
- No continuous monitoring (monitoring through inspections)
- At inspection time, if the device is not failed, it is replaced.
- Replacement occurs only after an inspection (in particular there is no replacement at times-to-failure)
- Replacement is AGAN
- c_r the replacement cost
- c_u the unavailability cost

Block replacement policy

- This policy depends on a single parameter: the delay between two consecutive inspections δ
- Aim: define the optimal inter-inspection delay $\delta^\star$
- The asymptotic cost per unit of time defined as follows:

$$C(\delta;\theta) = \lim_{t\to\infty} \frac{C_t(\delta)}{t},$$

- $C_t(\delta)$ is the cost over the time interval $[0, t]$ when the device is inspected at $(k\delta)_{k\in\mathbb{N}}$

Block replacement policy

According to the renewal theory, one has:

$$C(\delta;\theta) = \frac{\text{Expected cost over a cycle}}{\text{Expected cycle length}}.$$

For the block-replacement policy, we have:

$$C(\delta;\theta) = \frac{\mathbb{E}[c_r + c_u(\delta - T)_+]}{\delta} = \frac{c_r + c_u \int_0^\delta F_T(u;\theta)\mathrm{d}u}{\delta}.$$

Therefore,

$$\lim_{\delta\to 0} C(\delta;\theta) = +\infty \quad \text{and} \quad \lim_{\delta\to +\infty} C(\delta;\theta) = c_u.$$

Let us denote $\delta^\star := \mathrm{argmin}_{\delta>0} C(\delta;\theta)$. Differentiating the above expression of the cost function, $\delta^\star$ is the root of the following function (with respect to δ):

$$\phi(\delta;\theta) = \mathbb{E}[T\mathbf{1}_{T\leqslant\delta}] - \frac{c_r}{c_u}$$

where

$$\mathbb{E}[T\mathbf{1}_{T\leqslant\delta}] = \int_0^\delta u f_T(u;\theta)\mathrm{d}u = -\delta S_T(\delta;\theta) + \int_0^\delta S_T(u;\theta)\mathrm{d}u$$

Block replacement policy

Under this condition of existence for $\delta^\star$, it is well-known that the optimal cost is equal to:

$$C^\star = C(\delta^\star;\theta) = c_u F_T(\delta^\star;\theta).$$

It follows that the optimal delay is given by:

$$\delta^\star = F_T^{-1}(C^\star/c_u;\theta),$$

where $F_T^{-1}(\cdot;\theta)$ is the quantile function of the random variable T. Replacing this expression of the optimal delay in the function ϕ and after some simple algebra, one can obtain an implicit function ψ satisfied by $C^\star$:

$$\psi(C^\star;\theta) = \int_0^{C^\star/c_u} F_T^{-1}(u;\theta)\mathrm{d}u - \frac{c_r}{c_u} = 0.$$

Tools for the sensitivity analysis

- Let be $\theta_0 \in \Theta$ the true parameter of the time-to-failure distribution
- $\delta_0^\star$ can be computed by determining the root of $\phi(\cdot; \theta_0)$
- Let be $\widehat{\theta}_n$ an estimator of θ, for instance the maximum likelihood estimator
- consistency $\widehat{\theta}_n \xrightarrow[n\to\infty]{Pr} \theta_0$,
- asymptotic normality: $\sqrt{n}\left(\widehat{\theta}_n - \theta_0\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \Sigma_0)$,
- the asymptotic variance-covariance matrix Σ_0 depends on θ_0

Key: δ-method
when we want to replace θ_0 by $\widehat{\theta}_n$ in order to estimate $\delta_0^\star$

δ-method

Theorem (δ-method)

Let $(X_n)_{n\in\mathbb{N}^*}$ be a sequence of $\mathbb{R}^p$-valued random vectors. Assume there exists $\mu_X \in \mathbb{R}^p$ and Σ a definite positive matrix such that

$$\sqrt{n}\,(X_n - \mu_X) \xrightarrow[n\to\infty]{d} N(0,\Sigma).$$

Let q real functions $f_1,\ldots,f_q$ with continuous first partial derivatives at μ_X, where at least one of these derivatives is non-zero. For $i\in\{1,\ldots,q\}$ and for any $n\in\mathbb{N}^*$, set $Y_{i,n}=f_i(X_n)$, $Y_n=(Y_{1,n},\ldots,Y_{q,n})^T$ and $\mu_Y=(f_1(\mu_X),\ldots,f_q(\mu_X))^T$. Then, the sequence $(Y_n)_{n\in\mathbb{N}^*}$ is also asymptotically normal:

$$\sqrt{n}\,(Y_n - \mu_Y) \xrightarrow[n\to\infty]{d} N(0, K\Sigma K^T),$$

where K is the $q\times p$ matrix with elements $k_{i,j}=\partial f_i/\partial x_j$ for $i\in\{1,\ldots,p\}$ and $j\in\{1,\ldots,q\}$.

Extension of the δ-method

$\delta_0^\star$ and $C_0^\star$ are the solutions of implicit equations and δ-method not valid. Analogous tool for this situation proposed by Benichou and Gail:

Theorem (Benichou and Gail 1989)

Let $(X_n)_{n\in\mathbb{N}^*}$ be as in the previous theorem. Let $\mu_X \in \mathbb{R}^p$ and $\mu_Y \in \mathbb{R}^q$. Let $g_1, \ldots, g_q$ a set of q continuous functions from $\mathbb{R}^p \times \mathbb{R}^q$ into $\mathbb{R}$ with continuous first partial derivatives in an open set containing (μ_X, μ_Y). Let Y_n be the $\mathbb{R}^q$-valued random vectors satisfying $g_r(X_n, Y_n) = 0$ for all $r \in \{1, \ldots, q\}$. Let $J_{x,y}$ be the $q \times q$ matrix with elements $\frac{\partial g_i}{\partial y_j}(x, y)$ and let $H_{x,y}$ be the $q \times p$ matrix with elements $\frac{\partial g_i}{\partial x_j}(x, y)$. If $|J_{\mu_X,\mu_Y}| \neq 0$ and if each rows of $J^{-1}_{\mu_X,\mu_Y} H_{\mu_X,\mu_Y}$ contain at least one nonzero element, then

$$\sqrt{n}\,(Y_n - \mu_Y) \xrightarrow[n\to\infty]{d} N(0, J^{-1}_{\mu_X,\mu_Y} H_{\mu_X,\mu_Y} \Sigma H^T_{\mu_X,\mu_Y} (J^{-1}_{\mu_X,\mu_Y})^T).$$

Asymptotic behavior of the cost

Plug-in method: $\forall \delta > 0, \quad C(\delta; \widehat{\theta}_n) = \frac{c_r + c_u \int_0^\delta F_T(u; \widehat{\theta}_n) \mathrm{d}u}{\delta}$.

Theorem (simple application to the cost function)

Assume the hypotheses on $\widehat{\theta}_n$ are satisfied and that $\theta \mapsto C(\delta; \theta)$ is differentiable for any $\delta > 0$. Let $\nabla_\theta C(\delta; \theta)$ be the gradient vector of the cost function (with respect to θ). If $\nabla_\theta C(\delta; \theta)$ is continuous and if $\nabla_\theta C(\delta; \theta_0) \neq 0_{\mathbb{R}^p}$, then $C(\delta; \widehat{\theta}_n)$ is an asymptotic normal (point-wise) estimator of $C(\delta; \theta_0)$:

$$\forall \delta > 0, \quad \sqrt{n} \left(C(\delta; \widehat{\theta}_n) - C(\delta; \theta_0) \right) \xrightarrow[n \to \infty]{d} \mathcal{N}(0, \sigma^2_{cost}),$$

with $\sigma^2_{cost} = \nabla_\theta C(\delta; \theta_0) \Sigma_0 \nabla_\theta C(\delta; \theta_0)^T$.

Asymptotic behavior of the optimal inspection delay

Recall that $\delta^\star$ is the root of $\phi(\delta;\theta)$
As an application of the previous thoerems, we can also prove that $\widehat{\delta}_n^\star$ is also an asymptotically estimator of $\delta_0^\star$ under some regularity assumptions of the function ϕ.

Theorem (simple application to the optimal delay)

Assume that the hypotheses on $\widehat{\theta}_n$ are satisfied and that $\theta \mapsto \phi(\delta;\theta)$ is differentiable for any $\delta > 0$. Let $\nabla_\theta \phi(\delta;\theta)$ be the gradient vector of ϕ (with respect to θ). If $\nabla_\theta \phi(\delta;\theta)$ is continuous with $\nabla_\theta \phi(\delta;\theta_0) \neq 0_{\mathbb{R}^p}$ and if $f_T(\delta_0^\star;\theta_0) \neq 0$, then $\widehat{\delta}_n^\star$ is an asymptotic normal estimator of $\delta_0^\star$:

$$\sqrt{n}\left(\widehat{\delta}_n^\star - \delta_0^\star\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \sigma^2_{opt.delay}),$$

where $\sigma^2_{opt.delay}$ is given by:

$$\sigma^2_{opt.delay} = \frac{\nabla_\theta \phi(\delta_0^\star;\theta_0)\Sigma_0 \nabla_\theta \phi(\delta_0^\star;\theta_0)^T}{[\delta_0^\star f_T(\delta_0^\star;\theta_0)]^2}.$$

Asymptotic behavior of the optimal inspection cost

Theorem (simple application to the optimal cost)

Assume that the hypotheses on $\widehat{\theta}_n$ are satisfied and that $\theta \mapsto \psi(C^\star;\theta)$ is differentiable for any $C^\star > 0$. Let $\nabla_\theta \psi(C^\star;\theta)$ be the gradient vector of ψ (with respect to θ). If $\nabla_\theta \psi(C^\star;\theta)$ is continuous with $\nabla_\theta \psi(C_0^\star;\theta_0) \neq 0_{\mathbb{R}^p}$, then $\widehat{C}_n^\star$ is an asymptotic normal estimator of $C_0^\star$:

$$\sqrt{n}\left(\widehat{C}_n^\star - C_0^\star\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \sigma^2_{opt.cost}),$$

where $\sigma^2_{opt.cost}$ is given by:

$$\sigma^2_{opt.cost} = \frac{c_u^2 \nabla_\theta \psi(C_0^\star;\theta_0)\Sigma_0 \nabla_\theta \psi(C_0^\star;\theta_0)^T}{\left[F_T^{-1}(C_0^\star/c_u;\theta_0)\right]^2}.$$

The gradient function (with respect to θ) can also be expressed as follows:

$$\nabla_\theta \psi(C_0^\star;\theta_0) = \int_0^{C_0^\star/c_u} \nabla_\theta F_T^{-1}(u;\theta_0)\mathrm{d}u.$$

Exponential time to failure

- Assume that the time-to-failure T is exponentially distributed with unknown parameter $\lambda_0 \in \mathbb{R}_+ = \Theta$:
- $\widehat{\lambda}_n = \frac{n}{\sum_{i=1}^n T_i}$.
- $\sqrt{n}\left(\widehat{\lambda}_n - \lambda_0\right) \xrightarrow[n\to\infty]{d} \mathcal{N}(0, \lambda_0^2)$.
- The cost function:

$$\begin{aligned} C(\delta;\lambda_0) &= \frac{1}{\delta}\left\{c_r + c_u \int_0^\delta (1-\exp(-\lambda_0 u))\,\mathrm{d}u\right\} \\ &= \frac{1}{\delta}\left\{c_r + c_u\delta - \frac{c_u}{\lambda_0}\left(1 - e^{-\lambda_0\delta}\right)\right\}. \end{aligned}$$

- Optimal delay obtained by setting to zero

$$\partial_\lambda C(\delta;\lambda_0) = \frac{c_u}{\delta}\left(\frac{1}{\lambda_0^2} - \left(\frac{\delta}{\lambda_0} + \frac{1}{\lambda_0^2}\right)e^{-\lambda_0\delta}\right).$$

$$\sigma^2_{cost} = c_u^2\left[\frac{1}{\lambda_0\delta} - \left(1 + \frac{1}{\lambda_0\delta}\right)e^{-\lambda_0\delta}\right]^2.$$

Exponential time to failure

For such distribution, it turns to be:

$$\phi(\delta;\lambda_0) = \frac{1}{\lambda_0} - \left(\delta + \frac{1}{\lambda_0}\right) e^{-\lambda_0 \delta} - \frac{c_r}{c_u}.$$

The first order partial derivatives of ϕ are given by:

$$\partial_\delta \phi(\delta;\lambda_0) = \lambda_0 \delta e^{-\lambda_0 \delta}$$

$$\partial_\lambda \phi(\delta;\lambda_0) = -\frac{1}{\lambda_0^2} + \left(1 + \frac{1}{\lambda_0 \delta} + \frac{1}{\delta^2 \lambda_0^2}\right) \delta^2 e^{-\lambda_0 \delta}.$$

It follows that the asymptotic variance is equal to:

$$\sigma^2_{opt.delay} = \left(-\frac{e^{\lambda_0 \delta_0^\star}}{\delta_0^\star \lambda_0^2} + \left(1 + \frac{1}{\lambda_0 \delta_0^\star} + \frac{1}{{\delta_0^\star}^2 \lambda_0^2}\right) \delta_0^\star\right)^2$$

Exponential time to failure

At least, using the expression of the quantile function for the exponential distribution, we obtain that the optimal cost $C_0^\star$ satisfies the following equation:

$$\psi(C_0^\star;\lambda_0) = \left(1 - \frac{C_0^\star}{c_u}\right)\log\left(1 - \frac{C_0^\star}{c_u}\right) + \frac{C_0^\star}{c_u} - \lambda_0\frac{c_r}{c_u} = 0.$$

Then, the first order partial derivatives of ψ are given by:

$$\partial_\delta\psi(C^\star;\lambda_0) = -\frac{1}{c_u}\log\left(1 - \frac{C_0^\star}{c_u}\right)$$

and

$$\partial_\lambda\psi(C^\star;\lambda_0) = -\frac{c_r}{c_u}.$$

It follows that the asymptotic variance is equal to:

$$\sigma^2_{opt.cost} = \left[\frac{c_r\lambda_0^2}{\log\left(1 - \frac{C_0^\star}{c_u}\right)}\right]^2.$$

Weibull time to failure

- We consider a more general case by assuming that the time-to-failure T is Weibull distributed with parameter $\theta_0 = (\alpha_0, \beta_0) \in \Theta = \mathbb{R}_+^2$: The MLE are as follows:

$$\widehat{\alpha}_n = \left(\frac{1}{n} \sum_{i=1}^{n} T_i^{\widehat{\beta}_n} \right)^{\frac{1}{\widehat{\beta}_n}},$$

$$\widehat{\beta}_n = \left[\left(\sum_{i=1}^{n} T_i^{\widehat{\beta}_n} \log T_i \right) \left(\sum_{i=1}^{n} T_i^{\widehat{\beta}_n} \right)^{-1} - \frac{1}{n} \sum_{i=1}^{n} \log T_i \right]^{-1}$$

- Moreover

$$(\widehat{\alpha}_n, \widehat{\beta}_n) \xrightarrow[n \to \infty]{d} \mathcal{N}(0, \Sigma_0)$$

where

$$\Sigma_0 = \frac{1}{\alpha_0^2(\Psi_1(1) + \Psi(2)^2 - (1 + \Psi(1))^2)} \begin{pmatrix} (\alpha_0 \beta_0)^2 & -\alpha_0(1 + \Psi(1)) \\ -\alpha_0(1 + \Psi(1)) & \frac{\Psi_1(1) + \Psi(2)^2}{\beta_0^2} \end{pmatrix}$$

Weibull time to failure

$$
\begin{aligned}
C(\delta;\alpha_0,\beta_0) &= \frac{1}{\delta}\left\{c_r + c_u\int_0^\delta 1 - e^{\left(-\left(\frac{u}{\alpha_0}\right)^{\beta_0}\right)}\mathrm{d}u\right\}\\
&= \frac{1}{\delta}\left\{c_r + c_u\delta - c_u\frac{\alpha_0}{\beta_0}\gamma\left(\frac{1}{\beta_0},\left(\frac{\delta}{\alpha_0}\right)^{\beta_0}\right)\right\}.
\end{aligned}
$$

$$
\partial_\alpha C(\delta;\alpha_0,\beta_0) = \frac{1}{\delta}\left\{-\frac{c_u}{\beta_0}\gamma\left(\beta_0{}^{-1},\left(\frac{\delta}{\alpha_0}\right)^{\beta_0}\right) - c_u\left(\frac{\delta}{\alpha_0}\right)^{\beta_0}\left(\left(\frac{\delta}{\alpha_0}\right)^{\beta_0}\right)^{\beta_0{}^{-1}-1}\mathrm{e}^{-\left(\frac{\delta}{\alpha_0}\right)^{\beta_0}}\right\}
$$

$$
\partial_\beta C(\delta;\alpha_0,\beta_0) = \frac{c_u\,\alpha}{\delta\,\beta^2}\gamma\left(\beta^{-1},\left(\frac{\delta}{\alpha}\right)^\beta\right) - \frac{c_u\,\alpha}{\delta\,\beta^3}\gamma\left(\beta^{-1},\left(\frac{\delta}{\alpha}\right)^\beta\right)\ln\left(\left(\frac{\delta}{\alpha}\right)^\beta\right) \quad (1)
$$

$$
\frac{c_u\,\alpha}{\delta\,\beta^3}G_{2,3}^{3,0}\left(\left(\frac{\delta}{\alpha}\right)^\beta\,\middle|\,{}^{1,1}_{0,0,\beta-1}\right) - \frac{c_u\,\alpha}{\delta\,\beta}\left(\frac{\delta}{\alpha}\right)^\beta\ln\left(\frac{\delta}{\alpha}\right)\left(\left(\frac{\delta}{\alpha}\right)^\beta\right)^{\beta^{-1}-1}e^{-\left(\frac{\delta}{\alpha}\right)^\beta}. \quad (2)
$$

Therefore $\sigma^2_{opt.cost} = \nabla_\theta C(\delta;\theta_0)\Sigma_0\nabla_\theta C(\delta;\theta_0)^T$ can be easily calculated by replacing the partial derivatives and the variance matrix.

Weibull time to failure

Using the above identity for the finite integral of the survival function, we have:

$$\phi(\delta;\alpha_0,\beta_0) = -\delta\,\mathsf{exp}\left(-\left(\frac{\delta}{\alpha_0}\right)^{\beta_0}\right) + \frac{\alpha_0}{\beta_0}\gamma\left(\frac{1}{\beta_0},\left(\frac{\delta}{\alpha_0}\right)^{\beta_0}\right) - \frac{c_r}{c_u}.$$

Therefore, the first order partial derivatives of ϕ are as follows:

$$\begin{aligned}
\partial_\delta\phi(\delta;\alpha,\beta) &= -e^{-(\delta/\alpha)^\beta} + (\delta/\alpha)^\beta \beta e^{-(\delta/\alpha)^\beta} - \mathrm{e}^{-\left(\frac{t}{\alpha}\right)^\beta} + \left(\frac{t}{\alpha}\right)^\beta \beta\,\mathrm{e}^{-\left(\frac{t}{\alpha}\right)^\beta} \\
\partial_\alpha\phi(\delta;\alpha,\beta) &= -\frac{\delta\beta}{\alpha}\left(\frac{\delta}{\alpha}\right)^\beta \mathrm{e}^{-\left(\frac{\delta}{\alpha}\right)^\beta} + \frac{1}{\beta}\gamma\left(\frac{1}{\beta},\left(\frac{\delta}{\alpha}\right)^\beta\right) \\
\partial_\beta\phi(\delta;\alpha,\beta) &= \delta\left(\frac{\delta}{\alpha}\right)^\beta \ln\left(\frac{\delta}{\alpha}\right)\mathrm{e}^{-\left(\frac{\delta}{\alpha}\right)^\beta} - \frac{\alpha}{\beta^2}\gamma\left(\frac{1}{\beta},\left(\frac{\delta}{\alpha}\right)^\beta\right)
\end{aligned}$$

$$\sigma^2_{opt.delay} = \frac{\nabla_\theta\phi(\delta_0^\star;\theta_0)\Sigma_0\nabla_\theta\phi(\delta_0^\star;\theta_0)^T}{\left[\delta_0^\star f_T(\delta_0^\star;\theta_0)\right]^2}.$$

where $f(x;\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}\mathrm{e}^{-(x/\alpha)^\beta}$ is known

Weibull time to failure

For the Weibull distribution,

$$\forall t \geq 0, \quad F_T(t,\theta) = 1 - \exp\left(-\left(\frac{t}{\alpha_0}\right)^{\beta_0}\right).$$

Therefore $\delta^* = \alpha_0(-\ln(1 - C^*/c_u))^{1/\beta_0}$, where

$$\frac{\partial F_T^{-1}(C^*/c_u, \theta_0)}{\partial \alpha_0} = -\ln(1 - C^*/c_u)^{1/\beta_0}$$

and

$$\frac{\partial F_T^{-1}(C^*/c_u, \theta_0)}{\partial \beta_0} = \frac{\alpha_0 \ln(1 - C^*/c_u)^{1/\beta_0} \ln(\ln(1 - C^*/c_u))}{\beta_0^2}$$

Therefore we can calculate,

$$\sigma^2_{opt.cost} = \frac{c_u^2 \int_0^{C_0^\star/c_u} \nabla_\theta F_T^{-1}(u;\theta_0)\mathrm{d}u \Sigma_0 \int_0^{C_0^\star/c_u} \nabla_\theta F_T^{-1}(u;\theta_0)\mathrm{d}u^T}{\left[F_T^{-1}(C_0^\star/c_u;\theta_0)\right]^2}.$$

Thank you for your attention