Semiparametric inference in an imperfect maintenance model

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Outline

We consider a repairable system with times of failure

$$T_0 = 0 < T_1 < T_2 < \cdots < T_k < \cdots$$

where a corrective maintenance is carried out. As often, the efficiency of the maintenance lies between the two extreme cases: ABAO and AGAN.

Many models have been introduced in case of imperfect maintenance:

- BP (1983)
- Virtual age models of Kijima (1989)
- Geometric processes of Lam (1988) and its extension by Bordes & Mercier (2013)
- ARA and ARI models of Doyen & Gaudoin (2004)
- Among others...

The aim of this work is to introduce new (semiparametric) model(s) of imperfect maintenance as well as to drive statistical inference of its (euclidean and functional) parameters.

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Notations

Let

- $X_i = T_i - T_{i-1}$ be the ith interarrival times;
- $(N_t)_{t\in\mathbb{R}^+}$ be the associated counting process of the number of failures observed up to time t;
- $$\lambda_t = \lim_{\Delta t \to 0} \frac{1}{\Delta t} P(N_{(t+\Delta t)^-} - N_{t^-} = 1 | \mathcal{H}_{t^-})$$
  its failure intensity;
- $\lambda(t)$ is the hazard rate of the time T_1 before the first failure.

All of you know what are the:

Models of Reduction of Age or Intensity

Examples of Arithmetic Reduction are given by:

- the model of Arithmetic Reduction of virtual Age ARA_1 where the failure intensity is

$$\lambda_t = \lambda(t - \rho T_{N_{t-}});$$

- the model of Arithmetic Reduction of Intensity ARI_1 where the failure intensity is

$$\lambda_t = \lambda(t) - \rho\lambda(T_{N_{t-}}).$$

There exists also models of Geometric Reduction of the interarrival times

- Geometric process (GP) or Quasi-renewal process, introduced separately by Lam (1988) and Wang-Pham (1996), where the failure intensity is given by

$$\lambda_t = \rho^{N_{t-}} \lambda \left(\rho^{N_{t-}} (t - T_{N_{t-}}) \right)$$

- Extended Geometric process (EGP) introduced by Bordes & Mercier (2013) where the failure intensity is:

$$\lambda_t = \rho^{a(N_{t-}+1)} \lambda \left(\rho^{a(N_{t-}+1)} (t - T_{N_{t-}}) \right),$$

with $k \mapsto a(k)$ a known function of k: $a(k) = \sqrt{k-1}$ or $a(k) = \ln(k-1)$.

One can also consider models of Geometric Reduction of Age or Intensity. Idea first mentioned in Doyen & Gaudoin (2004) and reconsidered during their presentation in MMR (2015).

A first model of this kind has been introduced by Finkelstein (2008) where the failure intensity is given by

$$\lambda_t = \rho^{N_{t-}} \lambda(t - T_{N_{t-}}).$$

In this case the X_i are independent.

It seems that no statistical inference has been carried out on this model.

But this process shares the same kind of problem than the Geometric Process: strong decreasing of the interarrival times when ρ is not close to 1.

In order to dissipate this "explosion" phenomenon, one can adapt the idea of Bordes & Mercier (2013) in this case.

So we will consider and develop statistical inference for a model of Imperfect Maintenance with Geometric Reduction of Intensity where the failure intensity is given by

$$\lambda_t = \rho^{a(N_{t-}+1)} \lambda(t - T_{N_{t-}})$$

and $k \mapsto a(k)$ is a function supposed to be known. In this case also, the X_i are independent.

Outline

The model

Our model assumes that:

- we observe a single repairable system with recurrent failures;
- a Corrective Maintenance (CM) is performed after each failure;
- the efficiency of the CM is of two types
  - an Imperfect Maintenance with Geometric Reduction of Intensity is performed after the $N-1$ first failures

$$\lambda_t = \rho^{a(N_{t-}+1)}\lambda(t - T_{N_{t-}})$$

  - a perfect maintenance (AGAN) is performed after the Nth failure
  - and so on...
  - N is given but ρ and $\lambda(\cdot)$ not!

Comments.

- Our model can be seen as a generalization of the Finkelstein model in two directions: intensity and two different types of Maintenance.

- Our model shares some similarities with the model of Nakagawa (1988) which assumes that
  - Preventive Maintenance (PM) are performed at deterministic and periodic times
  - the efficiency of the PM follows a model of geometric reduction of intensity
  - A AGAN maintenance is performed after N PM
  - CM with ABAO efficiency are performed after each failure of the system

Outline

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Observations

Suppose that we observe n series of N interarrival times $X_1, \ldots, X_N$, where the $X_i = T_i - T_{i-1}$:

- are independent;
- and follow the Geometric Reduction of Intensity property introduced above, i.e. with respective cumulative risks functions $\Lambda_{X_1}, \ldots, \Lambda_{X_N}$ such that

$$\Lambda_{X_k}(\cdot) = \rho^{a(k)} \Lambda_{X_1}(\cdot), \tag{1}$$

  for $k = 1, \ldots, N$ and where $(a(k)_{k=1,\ldots,N})$ is a known series of reals.
- The real parameter ρ and the functional parameter $\Lambda_{X_1}(\cdot)$ are unknown and have to be estimated.

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From (1), we get the functional equations:

$$\ln \Lambda_{X_k}(\cdot) - \ln \Lambda_{X_1}(\cdot) - a(k)\gamma = 0, \qquad (2)$$

for $k = 2, \ldots, N$ and where $\gamma = \ln \rho$.

This enables us to derive two different estimators of γ.

First Estimator $\hat{\gamma}_1$

From (2), one can write:

$$\int_{\tau_1}^{\tau_2} (\ln \Lambda_{X_k}(t) - \ln \Lambda_{X_1}(t) - a(k)\gamma)^2 \, dt = 0,$$

for $k = 2, \ldots, N$ and where $\tau_1 > 0$ (resp. $\tau_2 > 0$) is “sufficiently” small (resp. “sufficiently” high).

An estimator of γ is given by:

$$\hat{\gamma}_1 = \arg\min_{\gamma} \sum_{k=2}^{N} \int_{\tau_1}^{\tau_2} \left(\ln \hat{\Lambda}_{X_k}(t) - \ln \hat{\Lambda}_{X_1}(t) - a(k)\gamma \right)^2 dt,$$

where the $\hat{\Lambda}_{X_k}(\cdot)$, for $k = 1, \ldots, N$, are the empirical estimators of the hazard rate functions.

One can show that:

$$\hat{\gamma}_1 = \frac{\sum_{k=2}^{N} a(k) \int_{\tau_1}^{\tau_2} \left(\ln \hat{\Lambda}_{X_k}(t) - \ln \hat{\Lambda}_{X_1}(t) \right) dt}{(\tau_2 - \tau_1) \sum_{k=2}^{N} a^2(k)}.$$

The case $a(k) = k - 1$ gives:

$$\hat{\gamma}_1 = \frac{6 \sum_{k=2}^{N} (k-1) \int_{\tau_1}^{\tau_2} \left(\ln \hat{\Lambda}_{X_k}(t) - \ln \hat{\Lambda}_{X_1}(t) \right) dt}{(\tau_2 - \tau_1) N(N-1)(2N-1)}.$$

Second Estimator $\hat{\gamma}_2$

But from (2) one can write another functional equation:

$$\sum_{k=2}^{N} \ln \Lambda_{X_k}(\cdot) - (N-1)\ln \Lambda_{X_1}(\cdot) - \gamma \sum_{k=2}^{N} a(k) \quad = \quad 0.$$

And another estimator of γ is given by:

$$\hat{\gamma}_2 = \arg\min_{\gamma} \int_{\tau_1}^{\tau_2} \left(\frac{1}{N-1} \sum_{k=2}^{N} \ln \hat{\Lambda}_{X_k}(t) - \ln \hat{\Lambda}_{X_1}(t) - \gamma \frac{\sum_{k=2}^{N} a(k)}{N-1} \right)^2 dt.$$

We have:

$$\hat{\gamma}_2 = \frac{\int_{\tau_1}^{\tau_2} \left(\frac{1}{N-1} \sum_{k=2}^{N} \ln \hat{\Lambda}_{X_k}(t) - \ln \hat{\Lambda}_{X_1}(t) \right) dt}{\frac{\tau_2 - \tau_2}{N-1} \sum_{k=2}^{N} a(k)},$$

and in the case $a(k) = k - 1$

$$\hat{\gamma}_2 = \frac{2 \int_{\tau_1}^{\tau_2} \left(\frac{1}{N-1} \sum_{k=2}^{N} \ln \hat{\Lambda}_{X_k}(t) - \ln \hat{\Lambda}_{X_1}(t) \right) dt}{(\tau_2 - \tau_1) N}.$$

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Theorem

When n tends to infinity, we have

$$\sqrt{n}(\hat{\gamma}_1 - \gamma) \xrightarrow{\mathcal{L}} N(0, \sigma_1^2)$$

and

$$\sqrt{n}(\hat{\gamma}_2 - \gamma) \xrightarrow{\mathcal{L}} N(0, \sigma_2^2),$$

where, e.g.

$$\sigma_1^2 = \frac{1}{\left[(\tau_2 - \tau_1)\sum_{k=2}^N a^2(k)\right]^2} \left\{ 2\sum_{k=2}^N a^2(k) \left(\iint_{\tau_1<s<t<\tau_2} \frac{\left(\frac{1}{\bar{F}_{X_1}(s)}\right)^{\beta^{a(k)}} - 1}{\beta^{2a(k)}\Lambda_{X_1}(s)\Lambda_{X_1}(t)} dsdt \right) + 2\left(\sum_{k=2}^N a(k)\right)^2 \left(\iint_{\tau_1<s<t<\tau_2} \frac{\frac{1}{\bar{F}_{X_1}(s)} - 1}{\Lambda_{X_1}(s)\Lambda_{X_1}(t)} dsdt \right) \right\},$$

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Simulation design[1]

- T_1 has a Weibull distribution with scale parameter $\alpha = 3$ and shape parameter $\beta = 0.8$.
- $\gamma \in \{0.5, 1, 1.5, 2\} \iff \rho \in \{1.65, 2.72, 4.48, 7.39\}$
- $a(k) = (k-1)/10$.
- Two scenarios:
  - Design 1: $N = 5$ and $n \in \{25, 50, 100, 200\}$
  - Design 2: $n = 100$ and $N \in \{3, 5, 7, 9\}$

[1]Simulations done by Youssef El Behi during his PFE

Outline

When n increases

n	γ	0,5	1	1,5	2
25	$\hat{\gamma}_1$	-0,03789	-0,04029	-0,03836	-0,03344
	$\hat{\gamma}_2$	-0,03946	-0,03263	-0,03271	-0,02829
50	$\hat{\gamma}_1$	-0,06343	-0,04936	-0,05080	-0,03353
	$\hat{\gamma}_2$	-0,06188	-0,04635	-0,04931	-0,02882
100	$\hat{\gamma}_1$	-0,05523	-0,04462	-0,03482	-0,02761
	$\hat{\gamma}_2$	-0,05227	-0,04069	-0,03010	-0,02406
200	$\hat{\gamma}_1$	-0,04806	-0,03577	-0,03432	-0,02555
	$\hat{\gamma}_2$	-0,04455	-0,03203	-0,03050	-0,02050

Table: Simulation results. Empirical Relative Bias of $\hat{\gamma}_1$ and $\hat{\gamma}_2$ when $T_1 \sim \text{Weib}(3, 0.8)$, $a(k) = (k-1)/10$ and $N = 5$.

n	γ	0,5	1	1,5	2
25	$\hat{\gamma}_1$	0,63095	0,69865	0,79370	0,88634
	$\hat{\gamma}_2$	0,88329	0,97972	1,12950	1,24687
50	$\hat{\gamma}_1$	0,29140	0,32147	0,34875	0,40675
	$\hat{\gamma}_2$	0,40546	0,45008	0,48642	0,57161
100	$\hat{\gamma}_1$	0,14421	0,15731	0,16352	0,18487
	$\hat{\gamma}_2$	0,20004	0,21735	0,22894	0,25575
200	$\hat{\gamma}_1$	0,07282	0,07691	0,08291	0,09043
	$\hat{\gamma}_2$	0,10033	0,10487	0,11257	0,12379

Table: Simulation results. Empirical Standard deviation of $\hat{\gamma}_1$ and $\hat{\gamma}_2$ when $T_1 \sim \text{Weib}(3, 0.8)$, $a(k) = (k-1)/10$ and $N = 5$.

n	γ	0,5	1	1,5	2
25	$\hat{\gamma}_1$	0,39846	0,48974	0,63327	0,79007
	$\hat{\gamma}_2$	0,78059	0,96092	1,27818	1,55789
50	$\hat{\gamma}_1$	0,08592	0,10578	0,12743	0,16994
	$\hat{\gamma}_2$	0,16536	0,20472	0,24208	0,33006
100	$\hat{\gamma}_1$	0,02156	0,02674	0,02947	0,03723
	$\hat{\gamma}_2$	0,04070	0,04890	0,05445	0,06772
200	$\hat{\gamma}_1$	0,00588	0,00719	0,00952	0,01079
	$\hat{\gamma}_2$	0,01056	0,01202	0,01477	0,01700

Table: Simulation results. Empirical Mean Squarre Error of $\hat{\gamma}_1$ and $\hat{\gamma}_2$ when $T_1 \sim \text{Weib}(3, 0.8)$, $a(k) = (k-1)/10$ and $N = 5$.

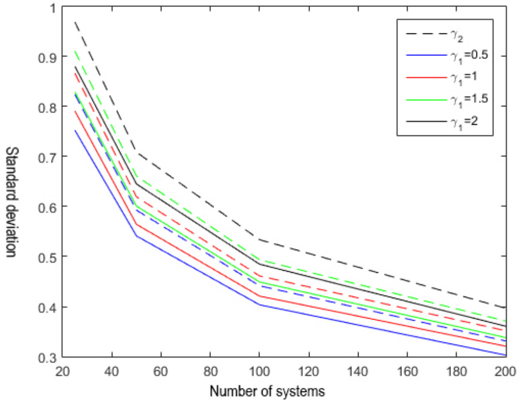

Figure: Simulation results. Empirical Standard deviation of $\hat{\gamma}_1$ and $\hat{\gamma}_2$ when $T_1 \sim \text{Weib}(3, 0.8)$, $a(k) = (k-1)/10$ and $N = 5$.

Outline

When N increases

N	γ	0,5	1	1,5	2
3	$\hat{\gamma}_1$	-0,06946	-0,05792	-0,04248	-0,03154
	$\hat{\gamma}_2$	-0,07218	-0,06350	-0,04249	-0,02677
5	$\hat{\gamma}_1$	-0,02051	-0,02749	-0,01609	-0,01454
	$\hat{\gamma}_2$	-0,01249	-0,02516	-0,01189	-0,01354
7	$\hat{\gamma}_1$	-0,02270	-0,01410	-0,01240	-0,00393
	$\hat{\gamma}_2$	-0,02353	-0,01345	-0,01159	-0,00195
9	$\hat{\gamma}_1$	-0,02199	-0,00839	-0,00398	-0,00181
	$\hat{\gamma}_2$	-0,02542	-0,00612	-0,00258	-0,00124

Table: Simulation results. Empirical Relative Bias of $\hat{\gamma}_1$ and $\hat{\gamma}_1$ when $T_1 \sim \text{Weib}(3, 0.8)$, $a(k) = (k-1)/10$ and $n = 100$.

N	γ	0,5	1	1,5	2
3	$\hat{\gamma}_1$	0,73164	0,72054	0,78019	0,79689
	$\hat{\gamma}_2$	0,86627	0,86339	0,94242	0,95786
5	$\hat{\gamma}_1$	0,20030	0,21327	0,23486	0,27570
	$\hat{\gamma}_2$	0,27995	0,30117	0,33059	0,39304
7	$\hat{\gamma}_1$	0,09221	0,11234	0,13231	0,15770
	$\hat{\gamma}_2$	0,13779	0,17054	0,20048	0,24185
9	$\hat{\gamma}_1$	0,05635	0,07325	0,08957	0,12031
	$\hat{\gamma}_2$	0,08781	0,11500	0,14194	0,19071

Table: Simulation results. Empirical Standard deviation of $\hat{\gamma}_1$ and $\hat{\gamma}_2$ when $T_1 \sim \text{Weib}(3, 0.8)$, $a(k) = (k-1)/10$ and $n = 100$.

N	γ	0,5	1	1,5	2
3	$\hat{\gamma}_1$	0,53650	0,52253	0,61276	0,63901
	$\hat{\gamma}_2$	0,75173	0,74947	0,89222	0,92036
5	$\hat{\gamma}_1$	0,04023	0,04624	0,05574	0,07686
	$\hat{\gamma}_2$	0,07841	0,09134	0,10961	0,15521
7	$\hat{\gamma}_1$	0,00863	0,01282	0,01785	0,02493
	$\hat{\gamma}_2$	0,01912	0,02926	0,04049	0,05851
9	$\hat{\gamma}_1$	0,00330	0,00544	0,00806	0,01449
	$\hat{\gamma}_2$	0,00787	0,01326	0,02016	0,03638

Table: Simulation results. Empirical Mean Squarre Error of $\hat{\gamma}_1$ and $\hat{\gamma}_2$ when $T_1 \sim \text{Weib}(3, 0.8)$, $a(k) = (k-1)/10$ and $n = 100$.

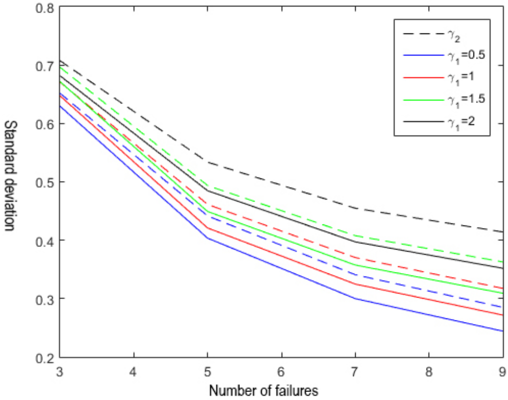

Figure: Simulation results. Empirical Standard deviation of $\hat{\gamma}_1$ and $\hat{\gamma}_2$ when $T_1 \sim \text{Weib}(3, 0.8)$, $a(k) = (k-1)/10$ and $n = 100$.

Outline

Application on Norsk Hydro data set

- Real dataset introduced by Bunea *et al.* (2003).
- Gives failure times of two identical compressor units of the Norsk Hydro ammonia plant between the 2nd of October 1968 and the 25th of June 1989.
- The dataset contains a large part of the history of the compressor units like: the time of component failure, the failure modes, the revision periods (18 revision periods with different lengths), etc...

We have decided to follow Bunea *et al.* (2003) and Dijoux and Gaudoin (2009) in considering the interarrival times as the operation times since last failure.

Like these authors, we have assumed that these times are i.i.d.

This yields 338 observed lifetimes without censoring.

We have considered a family of parametric functions for $a(k,\theta)$:

$$\mathcal{A} = \left\{ \left(\frac{k-1}{5}\right)^\theta ; \left(\frac{k-1}{10}\right)^\theta ; \left(\frac{k-1}{100}\right)^\theta ; (k-1)^\theta ; 2(k-1)^\theta ; 5(k-1)^\theta ; \theta \ln(k) ; \exp(k\theta) \right\}$$

and estimated the parametric family as well as the value of the parameters by

$$(\hat\gamma, \hat a(\cdot,\hat\theta)) = \arg \min_{\gamma, a(\cdot)\in\mathcal{A},\theta} \sum_{k=2}^{N} \int_{\tau_1}^{\tau_2} \left(\ln \hat\Lambda_{X_k}(t) - \ln \hat\Lambda_{X_1}(t) - a(k,\theta)\gamma \right)^2 dt.$$

We obtained $a(k,\theta) = \theta \ln k$, $\hat\theta = -0,65$ and $\hat\gamma = 0.5$.

Outline

Perspectives for further research

- Observation of one system on $[0, \tau]$ should be doable. How to deal with observation of n systems on interval $[0, \tau]$?
- Considering a parametric form for the function $a(k, \theta)$, where θ is unknown.
- Considering an Additive model rather than a multiplicative model.
- Improving the estimation of $\Lambda_{X_1}(\cdot)$ by taking into account the observation of the other interarrival times.
- Statistical tests on ρ or γ, Goodness-of-fit tests.

Thank you for your attention!