Optimal maintenance for minimal and imperfect repair models

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Outline

1. Repair, Maintenance and Replacement actions
2. Periodic maintenance (Barlow and Hunter, 1960)
3. Objective functions
4. Dynamic maintenance
5. The imperfect repair case
6. The minimal repair case
7. The Power Law Process case

Repair: Immediately after a failure (hence unscheduled)

- Minimal (as good as old)
- Imperfect
- Perfect (as good as new $\equiv$ replacement by a new equipment)

Maintenance: (Pre)scheduled

- Imperfect
- Perfect

Minimal repair $\Longrightarrow$ $N(t)$ is an NHPP with deterministic intensity $\lambda(t)$

Imperfect repair $\Longrightarrow$ $N(t)$ is a Counting Process with random intensity $\lambda(t)$

More precisely

- Under MR, there is a **deterministic** function $\lambda(t)$ such that

$$\lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}[N(t+h) - N(t)] = \lambda(t)\,.$$

- Under IR, there is a **random** process $\lambda(t)$ such that

$$\lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}[N(t+h) - N(t) | \mathscr{F}_t] = \lambda(t)\,,$$

where $\mathscr{F}_t$ denotes the history of failures up to t (e.g. $\mathscr{F}_t = \sigma\{N(s) : s \leq t\}$).

Alternatively,

- Under MR, $\mathbb{E}\, N(t) = \int_0^t \lambda(u)\, du = \Lambda(t)$;
- Under IR,

  $\mathbb{E}\, N(t) = \mathbb{E}\, \int_0^t \lambda(u)\, du = \int_0^t \mathbb{E}\, \lambda(u)\, du$ and
  $\phi(t) = \mathbb{E}\, \lambda(t)$ is the ROCOF function.

Periodic Maintenance (Barlow and Hunter, 1960)

- **Minimal** Repairs and **Perfect** Maintenance.
- System will be maintained at fixed and nonrandom time epochs τ (i.e. at τ, $2\,\tau$, $3\,\tau$ and so on).
- Instantaneous repair and maintenance actions (alternatively, down time of the system is included in the costs and time is actually *operating time*.
- Costs of repair and maintenance actions are random **but independent** of the failure history of the system.
- Expected cost of a repair action is 1; expected cost of a maintenance action is k.
- Intensity $\lambda(t)$ is increasing.

- Suppose the system will be operated during m maintenance cycles. Since the system is renewed at each maintenance, let $N_i(\tau)$ be the number of failures associated with each cycle.
- Let M_i be the cost of the i-th maintenance and R_{ij} the cost of the j-th repair inside the i-th maintenance.
- Total cost of the i-th cycle is

$$C_i = M_i + \sum_{j=1}^{N_i(\tau)} R_{ij}\,.$$

- Total cost for the m maintenance cycles will be

$$C = \sum_{i=1}^{m} C_i = \sum_{i=1}^{m} M_i + \sum_{i=1}^{m} \sum_{j=1}^{N_i(\tau)} R_{ij}\,.$$

- Cost *per unit of time* is

$$\overline{C} = \frac{C}{m\,\tau} = \frac{\sum_{i=1}^{m} M_i + \sum_{i=1}^{m} \sum_{j=1}^{N_i(\tau)} R_{ij}}{m\,\tau}$$

- Using the SLLN it is easy to see that

$$\lim_{m\to\infty} \overline{C} = \frac{k + EN(\tau)}{\tau} = \frac{k + \Lambda(\tau)}{\tau}$$

- However, it is easier to check that this is also the expected cost per unit of time for a **single** maintenance cycle.

To minimize

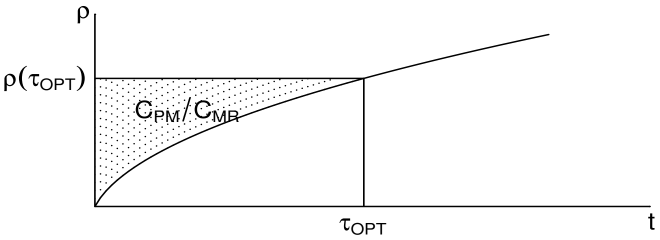
$$\overline{C}(\tau) = \mathbb{E}\,\frac{M_1 + \sum_{j=1}^{N_1(\tau)} R_{1j}}{\tau} = \frac{k + \Lambda(\tau)}{\tau}$$

is simple. For instance, we can differentiate to obtain that the optimal maintenance periodicity should satisfy that

$$\tau_P\,\lambda(\tau_P) - \Lambda(\tau_P) = k$$

or, defining $B(t) = t\,\lambda(t) - \Lambda(t)$,

$$\tau_P = B^{-1}(k)\,.$$


ρ
ρ(τOPT)
CPM/CMR
τOPT
t

Extensions

Imperfect Repair and Perfect Maintenance (e.g. Kijima et al, 1983; Toledo *et al.*, 2016):

- Substitute above $\lambda(t)$ and $\Lambda(t)$ by $\phi(t) = \mathbb{E}\,\lambda(t)$ and $\Phi(t) = \mathbb{E}\,\Lambda(t)$.
- Hence, define now $B(t) = t\,\phi(t) - \Phi(t)$ and compute the optimal periodicity as $\tau_P = B^{-1}(k)$.

Dynamic Optimization: If the information about the failure history of the system is available, why not use it to obtain a better solution?

- Since the history of the system is random, so will be the optimal maintenance time.
- Examples in the literature about *optimal replacement* (e.g. Aven, 1983, Aven and Bergman, 1986 or Lam, 1988, among others).

However, in the optimal replacement literature,

- Suppose that the system is to be maintained at random epochs τ_1 , $\tau_1 + \tau_2$ and so on.
- Assuming as above a large number m of maintenance actions, the cost per unit of time will be

$$\frac{C(\tau_1) + C(\tau_2) + \cdots + C(\tau_m)}{\tau_1 + \tau_2 + \cdots + \tau_m} = \frac{\tau_1 \overline{C}(\tau_1) + \cdots + \tau_m \overline{C}(\tau_m)}{\tau_1 + \cdots + \tau_m}$$

- Dividing by m numerator and denominator and taking limits this tends to

$$\frac{\mathbb{E}\, C(\tau)}{\mathbb{E}\, \tau} \neq \mathbb{E}\left[\frac{C(\tau)}{\tau}\right] = \mathbb{E}\, \overline{C}(\tau)$$

- Perfect maintenance and (possibly) imperfect repair
- Fixed costs of repair (1) and maintenance (k)
- *Continuous Wear-out* (Gilardoni *et al.*, 2016): We assume that the intensity $\lambda(t)$ is a **submartingale**, in the sense that $\mathbb{E}\left[\lambda(t)|\mathscr{F}_s\right] \geq \lambda(s)$.
- We want to find a random (i.e.stopping) time τ to minimize $\mathbb{E}\,\overline{C}(\tau)$.

Review of Counting Processes

- Probability space $(\Omega, \mathscr{F}, \mathbb{P})$
- Filtration $\{\mathscr{F}_t : t \geq 0\}$ (e.g. $\mathscr{F}_t = \sigma\{X(s) : s \leq t\}$).
- $N(t)$ is $\mathscr{F}_t$-msble
- $N(0) = 0$
- The maps $t \mapsto N(t)$ are right-continuous
- $dN(t) := N(t) - N(t-)$ equal either 0 or 1
- **Doob-Meyer Decomposition**: There exist a process $\lambda(t)$ and a martingale $M(t)$ such that

$$N(t) = \int_0^t \lambda(u)\, du + M(t)\,.$$

Main idea

- Note that for small $h \geq 0$ we have that $\mathbb{E}[N(t+h)|\mathscr{F}_t] \simeq N(t) + h\,\lambda(t)$
- Hence,

$$\mathbb{E}[\overline{C}(t+h) - \overline{C}(t)|\mathscr{F}_t] \\ = \mathbb{E}\left[\frac{k+N(t+h)}{t+h} - \frac{k+N(t)}{t}\right] \simeq \frac{h}{t+h}\left[\lambda(t) - \overline{C}(t)\right]$$

- This suggests to **stop** (i.e. maintain) the process whenever $\lambda(t) \geq \overline{C}(t)$.
- Define the **stopping time** $\tau = \inf\{t : \lambda(t) \geq \overline{C}(t)\}$

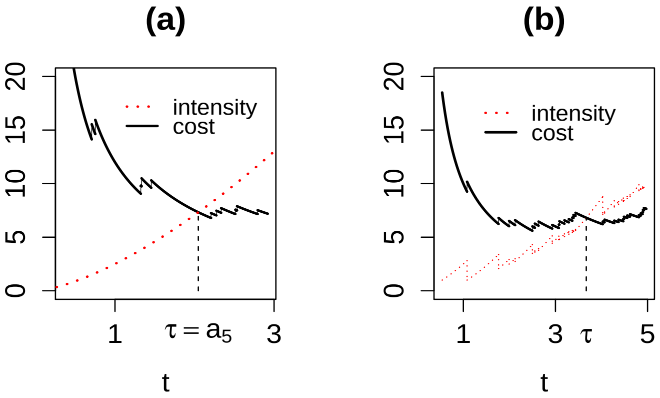


Figure : (a) Intensity function and realization of the cost per unit of time for a simulated realization of a PLP process with $\beta = 2.5$, $\theta = 1$ and $k = 10$ and (b) realization of the cost and intensity functions for a simulated realization of an ARA_1 process with the same initial PLP intensity and k and $\rho = 0.5$.

Main results

1. For any counting process $N(t)$ such that its intensity is a submartingale, we have that

$$\mathbb{E}\,\overline{C}(\tau) \leq \mathbb{E}\,\overline{C}(\tau_P) = \phi(\tau_P)\,. \tag{1}$$

2. Let $N(t)$ be an NHPP with an increasing intensity $\lambda(t)$ such that $\lim_{t\to\infty} \Lambda(t)/t = \infty$. Then, for any *stopping time* $\sigma \geq \tau$, $\mathbb{E}\,[\overline{C}(\sigma)\,|\,\mathscr{F}_\tau] \geq \overline{C}(\tau)$. Hence, for any *stopping time* σ (not necessarily greater than τ), $\mathbb{E}\,\overline{C}(\sigma) \geq \mathbb{E}\overline{C}(\tau)$.
3. If $N(t)$ is a PLP [i.e. $\lambda(t) = (\beta/\theta)\,(t/\theta)^{\beta-1}$], define $a_n = \eta\,[(k+n)/\beta]^{1/\beta}$. Then, for $n \geq 0$,

$$\mathbb{P}\,[\tau = a_n] = e^{-k\mu}\,k\,\frac{(n+k)^{n-1}}{n!}\,\mu^n\,e^{-n\mu}\,, \tag{2}$$

Outline of proofs

To prove the first result, we use integration by parts for *cadlag* processes:

$$f(b)g(b) - f(a)g(a) = \int_a^b f(u)\, dg(u) + \int_a^b g(u)\, df(u) + \sum_{a<u\leq b} \Delta g(u) \cdot \Delta f(u)\,.$$

Hence,

$$\overline{C}(t) - \overline{C}(s) = \int_s^t \frac{1}{u}\, d[k + N(u)] - \int_s^t \frac{k + N(u)}{u^2}\, du = \int_s^t \frac{A(u)}{u^2}\, du + \int_s^t \frac{1}{u}\, dM(u)\,,$$

where $A(t) = t\lambda(t) - k - N(t) = t\,[\lambda(t) - \overline{C}(t)]$. Next,

- The process $\int_s^t \frac{1}{u}\, dM(u)$ is a martingale and, using the Optional Sampling Theorem (OST), for any two stopping times τ and σ, $\mathbb{E} \int_\tau^\sigma \frac{1}{u}\, dM(u) = 0$.
- Hence,
$$\mathbb{E}[\overline{C}(\sigma) - \overline{C}(\tau)] = \mathbb{E} \int_\tau^\sigma \frac{A(u)}{u^2}\, du\,.$$
- Using the fact that $A(t) \leq 0$ if $t \leq \tau$, if we want to show that σ is worse than τ, it is enough to show that $\sigma \vee \tau$ is worse than τ.
- Using the fact that $\lambda(t)$ is a submartingale, it is easy to show that so should be $A(t)$.

Hence,

$$\mathbb{E}[\overline{C}(\tau_P \vee \tau) - \overline{C}(\tau)] = \mathbb{E} \int_{\tau}^{\tau_P \vee \tau} \frac{A(u)}{u^2}\, du$$

$$\mathbb{E}\left[\mathbb{E} \int_{\tau}^{\tau_P \vee \tau} \frac{A(u)}{u^2}\, du | \mathscr{F}_\tau\right]$$

$$= \mathbb{E}\left[\int_{\tau}^{\infty} \frac{\mathbb{E}[A(u) I(u \le \tau_P \vee \tau) | \mathscr{F}_\tau]}{u^2}\, du\right] \ge 0$$

because

$$\mathbb{E}[A(u) I(u \le \tau_P \vee \tau) | \mathscr{F}_\tau]$$

$$= \mathbb{E}[A(u) | \mathscr{F}_\tau]\, I(u \le \tau_P \vee \tau) \ge A(\tau)\, I(u \le \tau_P \vee \tau) \ge 0\,.$$

The proof of the second result is based on the following Lemma:

- Let $N(t)$ be an NHPP with mean function $\Lambda(t)$. Then the process $X(t) = \frac{N(t)}{\Lambda(t)}$ is a backwards martingale, i.e. for $s \leq t$, $N(s)|N(t) \sim$ Binomial $(n = N(t), p = \Lambda(s)/\Lambda(t))$ and

$$\mathbb{E}\left[\frac{N(s)}{\Lambda(s)}|N(t)\right] = \frac{1}{\Lambda(s)}\,N(t)\,\frac{\Lambda(s)}{\Lambda(t)} = \frac{N(t)}{\Lambda(t)}\,.$$

- Then, given $\mathscr{F}_t$, we use the strong markov property and this result for the process $X_\tau(t) = \frac{N(\tau+t)-N(\tau)}{\Lambda(\tau+t)-\Lambda(\tau)}$ and obtain a "backwards" stopping time which has the same expected cost as σ.

To find the distribution of τ for the PLP case, note first that τ is discrete in the NHPP case.

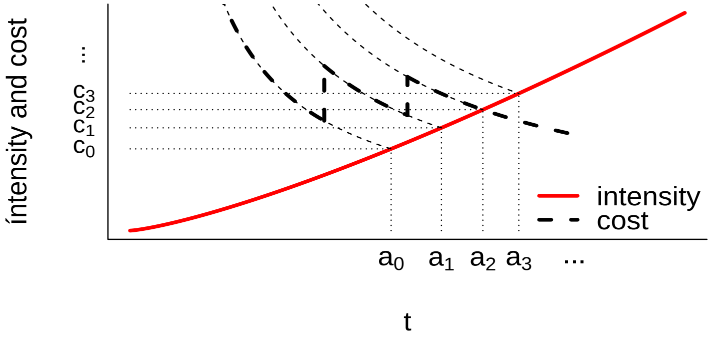


Figure : Admissible values of τ (a_n) and $\overline{C}(\tau)$ (c_n) in the NHPP case. Dashed curves are the maps $t \mapsto (k+n)/t$ ($n = 0, 1, \ldots$). Superimposed there is a trajectory of the cost for which $\tau = a_2$, i.e., the number of failures before maintenance is $N(\tau) = 2$.

- We use Wald's exponential mean one martingale

$$S_b(t) = \exp\{bN(t) - (e^b - 1)\,\Lambda(t)\}$$

- Next, we show that $\mathbb{E}\, S_b(\tau) = 1$.
- Substituting $N(\tau) = \tau\,\lambda(\tau) - k$ and noting that, in the PLP case, $\tau\,\lambda(\tau) = \beta\,(t/\theta)^{\beta} = \beta\,\Lambda(\tau)$, this allows us to compute the Moment Generating Function of τ.
- We note that the third result allows us to compute moments of τ and $\overline{C}(\tau)$ by summing series which converge fast.

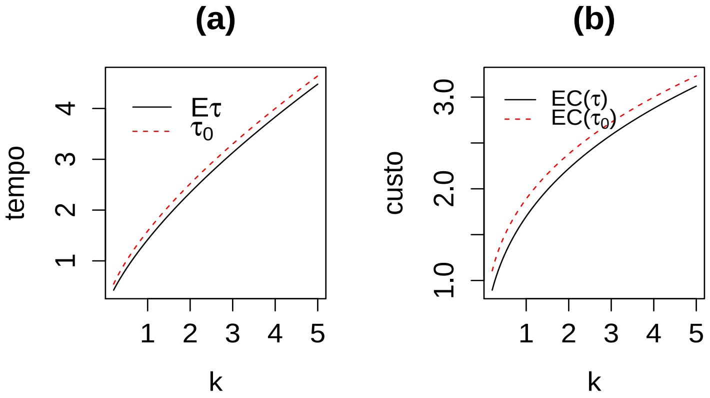


Figure : (a) Deterministic optimal time and expected optimal stopping time and (b) Expected cost using the optimal deterministic and optimal stopping time, both for a PLP process with scale $\theta = 1$ and shape parameter $\beta = 1.5$, against ratio of costs k.

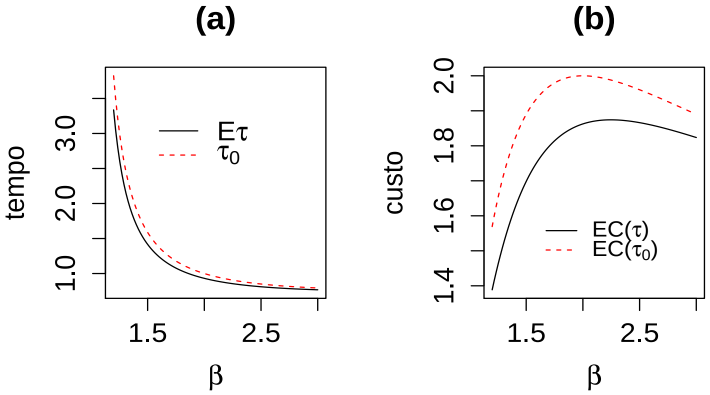


Figure : (a) Deterministic optimal time and expected optimal stopping time and (b) Expected cost using the optimal deterministic and optimal stopping time, both for a PLP process with scale $\theta = 1$ and ratio of costs $k = 1$, against form parameter β.

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