A revisit to the Problem of Stochastic Load Combination

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Outline

1. Motivation
   - The Basic Problem
   - Previous Work

2. Our Results
   - A single Load Process
   - Combination of Loads
   - Time above a given reference level

Outline

Load Combinations

- Sustained load, constant, varies rather slowly in time.
- Transient load, shocks which occur nearly as impulses.
- Linear load combination = sustained load + transient load.

Problem:

- First passage time distribution of the combined load
- Distribution of the maximum load combination over finite time intervals

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A Selection from the Literature

- Hasofer (1974)
  sustained load: Poisson Square Wave
  transient load: Marked Poisson process
- Larrabee & Cornell (1979) and (1981)
  load processes are modeled as filtered Poisson processes
  approximation based on the mean upcrossing rate
- Gaver & Jacbs (1981)
- Floris (1998)

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Square Wave Process

The square wave is used to model the sustained load.
Let $(T, X), (T_1, X_1), (T_2, X_2), \ldots$ be an iid sequence of random vectors with positive components.

- Define $S_0 \equiv 0$, $S_n = S_{n-1} + T_n$, $n \geq 1$.
- $N = \{N(t); t \geq 0\}$ is the counting process associated with the increasing sequence $0 < S_1 < S_2 < \ldots$

$$N(t) = \sum_{n=1}^{\infty} \mathbf{1}_{\{S_n \leq t\}}.$$

- The sustained load changes at the times S_n and X_n is the sustained load during the time interval $[S_{n-1}, S_n)$.

Maximum of the Square Wave Process

We assume that T and X are independent.
Let $X(t)$ be the sustained load at time t

$$X(t) = \sum_{n=1}^{\infty} X_n \mathbf{1}_{[S_{n-1}, S_n)}(t), \quad t \geq 0.$$

Then

$$X^*(t) = \max_{0 \leq s \leq t} X(s) = \max\left(X_1, \ldots, X_{N(t)}\right),$$

and

$$\mathbb{P}(X^*(t) \leq x) = F_X(x) \mathbb{E}\left[F_X(x)^{N(t)}\right].$$

A renewal-type Equation

If N is a homogeneous Poisson process with intensity λ, then

$$\mathbb{P}(X^*(t) \leq x) = F_X(x)e^{-\lambda \bar{F}_X(x)t}.$$

In general, $y(t) = \mathbb{E}\left[F_X(x)^{N(t)}\right]$ is a solution of the defective renewal equation

$$y(t) = \bar{F}_T(t) + F_X(x) \int_0^t y(t-s)\, \mathrm{d}F_T(s).$$

A renewal-type Equation, Sketch of Proof

$$X(t) = \sum_{n=1}^{\infty} X_n \mathbf{1}_{[S_{n-1}, S_n)}(t), \quad t \geq 0.$$

Define $\tilde{X}_n = X_{n+1}$, $\tilde{T}_n = T_{n+1}$, $\tilde{S}_n + \sum_{i=1}^{n} \tilde{T}_i$, and

$$\tilde{X}(t) = \sum_{n=1}^{\infty} \tilde{X}_n \mathbf{1}_{[\tilde{S}_{n-1}, \tilde{S}_n)}(t), \quad t \geq 0.$$

Then X and $\tilde{X}$ have the same distribution, and $\tilde{X}$ and (X_1, T_1) are independent.

$$X(t) = X_1 \mathbf{1}_{\{t < T_1\}} + \tilde{X}(t - T_1), \quad t \geq 0.$$

Shock Loads

- Shocks arrive according to a Poisson process $M = \{M(t); t \geq 0\}$ with intensity μ.
- the sequence of shocks $(Y_n)_{n\geq 1}$ is iid with CDF H.
- M and $Y_n)_{n\geq 1}$ are independent.

Maximum transient load is given by

$$Y^*(t) = \max\left(Y_1, \ldots, Y_{M(t)}\right)$$

and

$$\mathbb{P}(Y^*(t) \leq y) = e^{-\mu H(y) t}.$$

Superposition of Shock Loads

$Y_i = \{Y_i(t); t \geq 0\}$ is a shock load process with arrival intensity μ_i and shock size distribution H_i, $i = 1, \ldots, n$.
If $Y_1, \ldots, Y_n$ are independent, then the superposition $Y(t) = \sum_{i=1}^{n} Y_i(t)$, $t \geq 0$, is a shock load with arrival intensity $\mu = \mu_1 + \cdots + \mu_n$ and a shock size distribution with CDF

$$H(y) = \sum_{i=1}^{n} \frac{\mu_i}{\mu} H_i(y).$$

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Notation

- Sustained load X

$$X(t) = \sum_{n=1}^{\infty} X_n \mathbf{1}_{[S_{n-1}, S_n)}(t), \quad t \geq 0.$$

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- Transient load $Y = \{Y(t); t \geq 0\}$,
- X and Y are independent,
- Combined load

$$Z(t) = X(t) + Y(t), \quad t \geq 0.$$

Maximum Load over $[S_{n-1}, S_n)$

Define

$$\begin{aligned} U_n &= \max_{S_{n-1} \leq s < S_n} Z(s) \\ &= X_n + \max\left(Y_{M(S_{n-1})+1}, \ldots, Y_{M(S_n)}\right) \end{aligned}$$

Assume that the counting process N is Poisson with intensity λ. Then, the sequence $U_1, U_2, \ldots$ is iid with CDF

$$\mathbb{P}(U_n \leq x) = \int_0^x \frac{\mu}{\mu + \lambda H(x-y)} \, \mathrm{d}F_T(y).$$

CDF of $Z^*(t)$

$$\mathbb{P}(Z^*(t) \leq x) = \bar{F}_T(t)g(x,t) + \int_0^t g(x,s)\mathbb{P}(Z^*(t-s) \leq x)\,\mathrm{d}F_T(s),$$

with

$$g(x,t) = \int_0^x e^{-\mu \bar{H}(x-y)}\,\mathrm{d}F_X(y).$$

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Mean of the Total Time above a Reference Level x

$$
\begin{aligned}
L_x^+(t) &= \sum_{n=1}^{N(t)} T_n \mathbf{1}_{\{X_n > x\}} + \left(t - S_{N(t)}\right) \mathbf{1}_{\{X_{N(t)+1} > x\}} \\
&= \sum_{n=1}^{\infty} T_n \mathbf{1}_{\{X_n > x, S_n \le t\}} + (t - S_n) \mathbf{1}_{\{X_{N(t)+1} > x, S_n \le t < S_{n+1}\}}
\end{aligned}
$$

Define

$$
\phi(t) = \mathbb{E}\left[T_1 \mathbf{1}_{\{X_1 > x,\}}\right],
$$

then

$$
\mathbb{E}\left[L_x^+(t)\right] = \phi(t) + \int_0^t \phi(t-s)\, \mathrm{d}m_T(s).
$$

Probability distribution of $L_x^+(t)$

We assume that N is a Poisson process with intensity λ. The distribution of $L_x^+(t)$ has an atom in 0:

$$\mathbb{P}(L_x^+(t) = 0) = \mathbb{P}X^*(t) \leq x) = H(x)\mathbb{E}\left[H(x)^{N(t)}\right] = H(x)e^{-\lambda \bar{H}(x)t}.$$

We only present the case $t = S_n$.

$$\mathbb{P}(L_x^+(S_n) > u) = \sum_{k=1}^{n} B_p(n;k)\mathbb{P}(S_k > u),$$

with

$$B_p(n;k) = \binom{n}{k} p^k (1-p)^{n-k} \text{ and } p = \mathbb{P}(X > x).$$

Summary

- We derived explicit formulas for the probability distribution of combination of loads as a solution of a renewal-type equation.
- Our derivations also hold for general renewal processes, not only Poisson processes.
- We also have results for the total time above a reference level.