A bivariate random shock model for a two-component system with dependence

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Classical shock models

- Shocks arrival times:
  - (non) homogeneous Poisson process,
  - renewal process.
- Kinds of shocks:
  - extreme shock models (possible immediate failure),
  - cumulative shock models (increase of intrinsic characterisic: accumulated deterioration, failure rate, age, number of already endured shocks, ...),
  - mixed shock models.
- Possible dependence between:
  - arrival times and shock magnitudes,
  - arrival times and probability of system failure at shocks.
- Usually:
  - one single kind of intrinsic characterisic is considered for the system,
  - one single type of dependence is considered.

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The model (I)

- Two-component series system.
- Intrinsic characteristics:
  - first component: failure rate $h(t)$,
  - second component: accumulated (non negative) deterioration $(G_t)_{t\geq 0}$, failure threshold L.
- Mixed shock model:
  - arrival times: T_1, ..., T_n, ...; non homogeneous Poisson process $(N_t)_{t\geq 0}$ with intensity $d\Lambda(x) = \lambda(x)\ dx$,
  - probability for a shock at time T_n not to be fatal (Bernoulli trial): $q(T_n)$,
  - increment of failure rate (first component) at time T_n: $V_n^{(1)}$; failure rate at time t:

$$X_t^{(1)} = h(t) + \underbrace{\sum_{n=1}^{N_t} V_n^{(1)}}_{A_t^{(1)}},$$

  - increment of deterioration (second component) at time T_n: $V_n^{(2)}$; deterioration at time t:

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The model (II)

- Assumptions:
  - $V_n = \left(V_n^{(1)}, V_n^{(2)}\right)$ are i.i.d and independent of $(N_t)_{t \geq 0}$, so that

$$A_t = \left(A_t^{(1)}, A_t^{(2)}\right) = \sum_{n=1}^{N_t} V_n$$

    is a bivariate compound non homogeneous Poisson process,
  - fatality of a shock at time T_n (with probability $1 - q(T_n)$): depends on all other things only through T_n,
  - both components are conditionaly independent given $(\mathcal{F}_t)_{t \geq 0}$, where $\mathcal{F}_t = \sigma(A_t, t \geq 0)$.
- Stochastic dependence between components:
  - simultaneous shocks on both components,
  - dependence between increments of failure rate (first component) and deterioration (second component),
  - fatality of a shock: identical for both components,
- Other dependence:
  - arrival time and fatality of a shock are correlated.

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J. H. Cha and M. Finkelstein. On a terminating shock process with independent wear increments. *Journal of Applied Probability*, **46**(2):353–362, 2009.

J. H. Cha and J. Mi. Study of a stochastic failure model in a random environment. *Journal of Applied Probability*, **44**(1):151–163, 2007.

J. H. Cha and J. Mi. On a stochastic survival model for a system under randomly variable environment. *Methodology and Computing in Applied Probability*, **13**(3): 549–561, 2011.

J. D. Esary, A. W. Marshall, and F. Proschan. Shock models and wear processes. *The Annals of Probability*, **1**(4):627–649, 1973.

C. Qian, S. Nakamura, and T. Nakagawa. Cumulative damage model with two kinds of shocks and its application to the backup policy. *Journal of the Operations Research Society of Japan-Keiei Kagaku*, **42**(4):501–511, 1999.

System lifetime

- System lifetime:

$$\tau = \min\left(\tau_1, \tau_2, \tau_3\right).$$

- Components intrinsic failure times on $[0, t]$:

$$\mathbb{P}\left(\tau_1 > t | \mathcal{F}_t\right) = e^{-\int_0^t X_s^{(1)}\, ds} = e^{-H(t)} e^{-\int_0^t A_s^{(1)}\, ds} = e^{-H(t)} e^{-\sum_{i=1}^{N_t} (t - T_i) V_i^{(1)}}$$

$$\mathbb{P}\left(\tau_2 > t | \mathcal{F}_t\right) = \mathbb{P}\left(X_t^{(2)} \leq L | \mathcal{F}_t\right) = \mathbb{P}\left(G_t + A_t^{(2)} \leq L | \mathcal{F}_t\right) = F_{G_t}\left(L - A_t^{(2)}\right)$$

where

$$H\left(t\right) = \int_0^t h(s)\, ds.$$

- Time to the first fatal shock:

$$\mathbb{P}\left(\tau_3 > t | \mathcal{F}_t\right) = \prod_{i=1}^{N_t} q\left(T_i\right).$$

τ_1, τ_2 and τ_3 are conditionnaly independent given $\mathcal{F}_t$.

Computation of the reliability (I)

Proposition

The reliability is given by

$$R_L(t) = \mathbb{P}(\tau > t) = e^{-H(t)} \phi_t(L),$$

with

$$\phi_t(L) = \mathbb{E}\left[e^{-\sum_{i=1}^{N_t}(t-T_i)V_i^{(1)}} F_{G_t}\left(L - \underbrace{\sum_{i=1}^{N_t} V_i^{(2)}}_{A_t^{(2)}} \right) \prod_{i=1}^{N_t} q(T_i) \right].$$

Proof.

$$\begin{aligned} R_L(t) &= \mathbb{P}(\tau_1 > t, \tau_2 > t, \tau_3 > t) \\ &= \mathbb{E}\left[\mathbb{E}\left(\mathbf{1}_{\{\tau_1 > t, \tau_2 > t, \tau_3 > t\}} | \mathcal{F}_t \right) \right] \\ &= \mathbb{E}\left[\mathbb{E}\left(\mathbf{1}_{\{\tau_1 > t\}} | \mathcal{F}_t \right) \mathbb{E}\left(\mathbf{1}_{\{\tau_2 > t\}} | \mathcal{F}_t \right) \mathbb{E}\left(\mathbf{1}_{\{\tau_3 > t\}} | \mathcal{F}_t \right) \right] \\ &= e^{-H(t)} \phi_t(L). \end{aligned}$$

Computation of the reliability (II)

Theorem

The Laplace transform of $\phi_t(L)$ with respect to L is given by

$$\tilde{\phi}_t(s) = \tilde{F}_{G_t}(s)\tilde{\nu}_t(s)$$

where

$$\tilde{\nu}_t(s) = e^{-\Lambda(t)+((q\lambda)*\tilde{\mu}(\cdot,s))(t)}$$

and

- $\tilde{\mu}$ *is the bivariate Laplace transform of the distribution* μ *of* $V = \left(V^{(1)}, V^{(2)}\right)$:

$$\tilde{\mu}(u,s) = \iint_{\mathbb{R}_+^2} e^{-uv_1-sv_2}\mu(dv_1,dv_2),\ \textit{all}\ u,s \geq 0,$$

- $\tilde{\mu}(\cdot,s) : u \to \tilde{\mu}(u,s)$, *all* $s \geq 0$.

Proof.
We have:

$$\begin{aligned}\tilde{\phi}_t(s) &= \int_0^\infty e^{-sL}\ \mathbb{E}\left[F_{G_t}\left(L - A_t^{(2)}\right) e^{-\sum_{i=1}^{N_t}(t-T_i)V_i^{(1)}} \prod_{i=1}^{N_t} q(T_i)\right]\ dL \\ &= \mathbb{E}\left[\left(\int_0^\infty e^{-sL} F_{G_t}\left(L - A_t^{(2)}\right)\ dL\right) e^{-\sum_{i=1}^{N_t}(t-T_i)V_i^{(1)}} \prod_{i=1}^{N_t} q(T_i)\right]\end{aligned}$$

with

$$\begin{aligned}&\int_0^\infty e^{-sL} F_{G_t}\left(L - A_t^{(2)}\right)\ dL \\ &= e^{-sA_t^{(2)}} \tilde{F}_{G_t}(s) \\ &= e^{-s\sum_{i=1}^{N_t} V_i^{(2)}} \tilde{F}_{G_t}(s)\end{aligned}$$

(easy computation).

This provides:

$$\tilde{\phi}_t(s) = \tilde{F}_{G_t}(s)\,\theta(s)$$

with

$$\begin{aligned}\theta(s) &= \mathbb{E}\left[e^{-\sum_{i=1}^{N_t}(t-T_i)V_i^{(1)}}e^{-s\sum_{i=1}^{N_t}V_i^{(2)}}\prod_{i=1}^{N_t}q(T_i)\right] \\ &= \mathbb{E}\left[e^{-\sum_{i=1}^{+\infty}\left((t-T_i)V_i^{(1)}+sV_i^{(2)}-\ln q(T_i)\right)\mathbf{1}_{\{T_i\leq t\}}}\right] \\ &= \mathbb{E}\left(e^{-\sum_{i=1}^{\infty}\psi_{s,t}(V_i^{(1)},V_i^{(2)},T_i)}\right) \\ &= \mathbb{E}\left(e^{-M\psi_{s,t}}\right)\end{aligned}$$

where

$$\psi_{s,t}(v_1,v_2,w) = ((t-w)v_1 + sv_2 - \ln q(w))\,\mathbf{1}_{\{w\leq t\}}$$

and

$$M = \sum_i \delta_{(V_i^{(1)},V_i^{(2)},T_i)}$$

is a Poisson random measure with intensity measure

$$\nu(dv_1,dv_2,dw) = \mu(dv_1,dv_2)\lambda(w)dw.$$

Based on the formula for functionnal Laplace transforms of Poisson random measures:

$$\theta(s) = \exp\left(- \iiint_{\mathbb{R}_+^3} \left(1 - e^{-\psi_{s,t}}\right) d\nu \right)$$

with

$$\begin{aligned} & \iiint_{\mathbb{R}_+^3} \left(1 - e^{-\psi_{s,t}}\right) d\nu \\ & = \iiint_{\mathbb{R}_+^3} \left(1 - e^{-\psi_{s,t}(v_1,v_2,w)}\right) \mu(dv_1, dv_2)\lambda(w)dw \\ & = \Lambda(t) - \left[(q\lambda) * (\tilde{\mu}(\cdot, s))\right](t) \end{aligned}$$

(easy computation).

Erhan Çinlar. *Probability and stochastics*, volume 261 of *Graduate texts in Mathematics*. Springer Science + Business Media, 2011.

New Better than Used property

Theorem

Assume that:

- $e^{-H(s)}$ *is NBU,*
- $F_{G_{t+s}} \leq F_{G_t} F_{G_s}$ *(OK if* $(G_t)_{t\geq 0}$ *is a univariate subordinator).*

Then τ *is NBU as soon as one of two following conditions is satisfied:*

1. *q is non increasing and* λ *is constant,*
2. *q is constant and* Λ *is super-additive.*

Proof.
The point: show that

$$\phi_{t+s}(L) \leq \phi_t(L)\phi_s(L)$$

with

$$\phi_{s+t}(L) = \mathbb{E}\left[F_{G_{t+s}}\left(L - \sum_{i=1}^{N_{t+s}} V_i^{(2)}\right) e^{-\sum_{i=1}^{N_{t+s}}(t+s-T_i)V_i^{(1)}} \prod_{i=1}^{N_{t+s}} q(T_i)\right].$$

Using $F_{G_{t+s}} \leq F_{G_t}F_{G_s}$, $N_t \leq N_{t+s}$ and non increasingness of q, we get:

$$\phi_{s+t}(L) \leq \mathbb{E}\Bigg[F_{G_t}\left(L - \sum_{i=1}^{N_t} V_i^{(2)}\right) e^{-\sum_{i=1}^{N_t}(t-T_i)V_i^{(1)}} \prod_{i=1}^{N_t} q(T_i)$$

$$\times \underbrace{F_{G_s}\left(L - \sum_{i=N_t+1}^{N_{t+s}} V_i^{(2)}\right) e^{-\sum_{i=N_t+1}^{N_{t+s}}(s-(T_i-t))V_i^{(1)}} \prod_{i=N_t+1}^{N_{t+s}} q(T_i - t)}_{F_{G_s}\left(L - \sum_{j=1}^{N_s^{(t)}} V_{j+N_t}{}^{(2)}\right) e^{-\sum_{j=1}^{N_s^{(t)}}(s-T_j^{(t)})V_{j+N_t}{}^{(1)}} \prod_{j=1}^{N_s^{(t)}} q(T_j^{(t)})}\Bigg]$$

where

$$j = N_t - i, \quad N_s^{(t)} := N_{t+s} - N_t \text{ and } T_j^{(t)} := T_{N_t+j} - t.$$

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$$\left. \times \underbrace{F_{G_s}\left(L - \sum_{i=N_t+1}^{N_{t+s}} V_i^{(2)}\right) e^{-\sum_{i=N_t+1}^{N_{t+s}}(s-(T_i-t))V_i^{(1)}} \prod_{i=N_t+1}^{N_{t+s}} q(T_i - t)}_{F_{G_s}\left(L - \sum_{j=1}^{N_s^{(t)}} V_{j+N_t}{}^{(2)}\right) e^{-\sum_{j=1}^{N_s^{(t)}}(s-T_j^{(t)})V_{j+N_t}{}^{(1)}} \prod_{j=1}^{N_s^{(t)}} q(T_j^{(t)})}\right]$$

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We get:

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$$\left. \times\ \mathbb{E}\left(F_{G_s}\left(L - \sum_{j=1}^{N_s^{(t)}} V_{j+N_t}{}^{(2)}\right) e^{-\sum_{j=1}^{N_s^{(t)}}(s-T_j^{(t)})V_{j+N_t}{}^{(1)}} \prod_{j=1}^{N_s^{(t)}} q(T_j^{(t)}) \middle| \mathcal{F}_t\right)\right]$$

Based on the independent increments of $(N_t)_{t\geq 0}$:

$$\phi_{s+t}(L) \leq \phi_t(L) \times \phi_s^{(t)}(L)$$

where

$$\phi_s^{(t)}(L) = \mathbb{E}\left[F_{G_s}\left(L - \sum_{j=1}^{N_s^{(t)}} V_j{}^{(2)}\right) e^{-\sum_{j=1}^{N_s^{(t)}}(s-T_j^{(t)})V_j{}^{(1)}} \prod_{j=1}^{N_s^{(t)}} q(T_j^{(t)})\right].$$

The point now: show that $\phi_s^{(t)}(L) \leq \phi_s(L)$.

- If λ is constant: $\phi_s^{(t)}(L) = \phi_s(L)$.
- If q is constant and Λ is supper-additive:

$$\Lambda(s+t) - \Lambda(t) \geq \Lambda(s),$$

we use the fact that

$$\left(T_n^{(t)}\right)_{n\geq 0} \leq_{st} (T_n)_{n\geq 0}$$

to conclude that $\phi_s^{(t)}(L) \leq \phi_s(L)$.

M. Shaked and J. G. Shanthikumar. *Stochastic orders*. Springer Series in Statistics. Springer, 2006.

Influence of the model parameters

Theorem

Two different systems with identical parameters except from one parameter.

1. *If* $q(w) \leq \tilde{q}(w)$*, then* $\tau \leq_{st} \tilde{\tau}$*.*
2. *If* $\Lambda \geq \tilde{\Lambda}$ *and* q *is non decreasing, then* $\tau \leq_{st} \tilde{\tau}$*.*
3. *If* $V \leq_{lo} \tilde{V}$ *(namely* $F_V \leq F_{\tilde{V}}$*), then* $\tau \leq_{st} \tilde{\tau}$*.*

Conclusion

- Conditions for τ to be IFRA, IFR, DMRL???
- The lifetime is all the shorter as Λ is larger: only under the condition on non decreasingness of q. Can we remove this condition???
- Development of statistical estimation procedure???
- Study of condition-based maintenance policies???